2021/22

QUIZ 1

1. (a) Write the precise or formal (i.e., $\varepsilon - \delta$) definition of the following limit statement

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

Hence, using the above definition show that $\lim_{(x,y)\to(1,1)} xy + y^2 = 2$.

(b) Using different paths or iterated limit approach, find $\lim_{(x,y)\to(0,0)} \frac{3x^2y^2}{x^2+y^2}$ and verify using $\varepsilon - \delta$ definition of limit of a function.

2. (a) i. Determine and sketch the domain of the function

$$\ln[(16 - x^2 - y^2)(x^2 + y^2 - 4)].$$

ii. Evaluate the indicated limit or explain why it does not exist

$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2 + y^2}$$

(b) How can the function

$$f(x,y) = \frac{x^3 - y^3}{x - y}, \text{if } x \neq y$$

be re-defined along the line $x = y$ so that the resulting function is continuous on the whole xy-plane.

SOLUTION

1. (a) Given $\varepsilon > 0, \exists\, \delta_\varepsilon > 0$ such that whenever $|x - a| < \delta$ and $|y - b| < \delta$ holds then $|f(x,y) - L| < \varepsilon$

Now, showing that $\lim_{(x,y)\to(1,1)} xy + y^2 = 2$
Given $\varepsilon > 0, \exists\, \delta_\varepsilon > 0$ such that whenever $|x - 1| < \delta$ and $|y - 1| < \delta$ holds then $|xy + y^2 - 2| < \varepsilon$

$$\begin{aligned}
|xy + y^2 - 2| &= |xy - y + y + y^2 - 1 - 1| \\
&= |y(x-1) + (y-1) + (y^2 - 1)| \\
&= |y(x-1) + (y-1) + (y-1)(y+1)| \\
&\leq |y||x-1| + |y-1| + |y-1||y+1| \\
&= |y - 1 + 1||x-1| + |y-1| + |y-1||y - 1 + 2| \\
&= (|y-1| + 1)|x-1| + |y-1| + |y-1|(|y-1| + 2)
\end{aligned}$$

$$< (\delta+1)\delta + \delta + \delta(\delta+2)$$
$$= \delta^2 + \delta + \delta + \delta^2 + 2\delta$$
If $\delta \le 1$
$$< \delta + \delta + \delta + \delta + 2\delta$$
$$= 6\delta$$
$$< \varepsilon \quad \text{if } \delta = \frac{\varepsilon}{6}$$

By choosing $\delta = \min\left\{1, \frac{\varepsilon}{6}\right\}$ we can see that whenever $|x-1| < \delta$ and $|y-1| < \delta$ holds then $|xy + y^2 - 2| < \varepsilon$

Therefore, $\lim_{(x,y)\to(1,1)} xy + y^2 = 2$

(b) Given $\lim_{(x,y)\to(0,0)} \frac{3x^2y^2}{x^2+y^2}$, let $f(x,y) = \frac{3x^2y^2}{x^2+y^2}$

Using different paths approach

Consider along the path $y = mx$

$$f(x,mx) = \frac{3x^2(mx)^2}{x^2+(mx)^2}$$
$$= \frac{3x^2 \cdot m^2 \cdot x^2}{x^2 + m^2 \cdot x^2}$$
$$= \frac{3x^4m^2}{x^2(1+m^2)}$$
$$= \frac{3x^2m^2}{(1+m^2)}$$
$$\Rightarrow \lim_{x\to 0} \frac{3x^2m^2}{(1+m^2)} = \frac{3(0)^2m^2}{(1+m^2)} = 0$$

Consider along the path $x = 0$

$$f(0,y) = \frac{3(0)^2y^2}{0^2+y^2}$$
$$= \frac{0}{y^2} = 0$$
$$\Rightarrow \lim_{y\to 0} 0 = 0$$

Consider along the path $y = x^2$

$$f(x,x) = \frac{3x^2(x^2)^2}{x^2+(x^2)^2}$$
$$= \frac{3x^2 \cdot x^4}{x^2 + x^4}$$
$$= \frac{3x^6}{x^2(1+x^2)}$$

$$= \frac{3x^4}{1+x^2}$$
$$\Rightarrow \lim_{x\to 0} \frac{3(0)^4}{(1+0^2)} = 0$$

Since, different path approach have the same limit, we suspect that $\lim_{(x,y)\to(0,0)} \frac{3x^2y^2}{x^2+y^2} = 0$

Verifying using $\varepsilon - \delta$ definition of limit
Given $\varepsilon > 0, \exists\, \delta_\varepsilon > 0$ such that whenever $|x-0| < \delta$ and $|y-0| < \delta$ holds then
$\left|\frac{3x^2y^2}{x^2+y^2} - 0\right| < \varepsilon$

$$\left|\frac{3x^2y^2}{x^2+y^2} - 0\right| = \left|\frac{3x^2y^2}{x^2+y^2}\right|$$
$$= \left|3x^2 \cdot \frac{y^2}{x^2+y^2}\right|$$
$$\leq 3|x^2| \cdot \frac{y^2}{x^2+y^2}$$
$$\leq 3|x^2| \cdot (1) \quad \text{since, } \frac{y^2}{x^2+y^2} < 1$$
$$= 3|x-0|^2$$
$$< 3\delta^2$$
If $\delta \leq 1$
$$\leq 3\delta$$
$$< \varepsilon \quad \text{if } \delta = \frac{\varepsilon}{3}$$

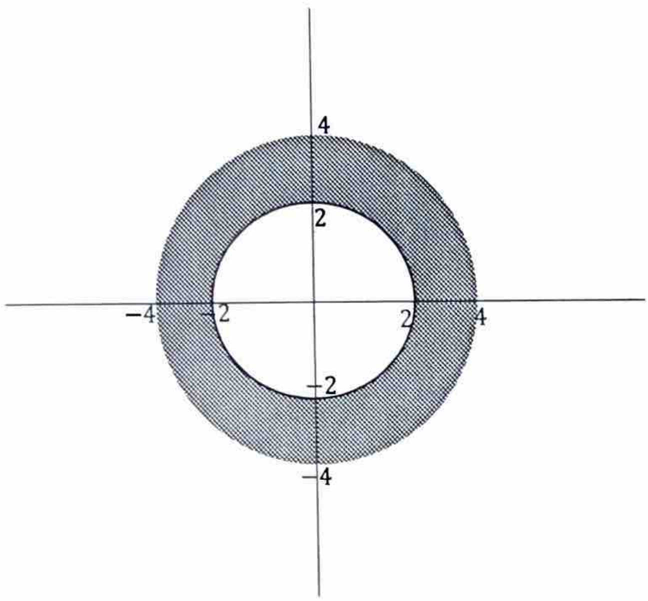
By choosing $\delta = \min\left\{1, \frac{\varepsilon}{3}\right\}$ we can see that whenever $|x-0| < \delta$ and $|y-0| < \delta$ holds then $\left|\frac{3x^2y^2}{x^2+y^2} - 0\right| < \varepsilon$

Therefore, $\lim_{(x,y)\to(0,0)} \frac{3x^2y^2}{x^2+y^2} = 0.$

2. (a) (i) Given $\ln[(16-x^2-y^2)(x^2+y^2-4)] = \ln(16-x^2-y^2) + \ln(x^2+y^2-4)$
$$D_f = \{(x,y) \in \mathbb{R} : 16-x^2-y^2 > 0,\ x^2+y^2-4 > 0\}$$
$$= \{(x,y) \in \mathbb{R} : x^2+y^2 < 16,\ x^2+y^2 > 4\}$$

Sketching the domain,
Let $x^2+y^2 = 16$
$x^2+y^2 = 4^2$
Thus, the domain is a circle with center $(0,0)$ and radius of 4.

Similarly,
Let $x^2 + y^2 = 4$
$x^2 + y^2 = 2^2$
Thus, the domain is a circle with center (0,0) and radius of 2.



Therefore, the domain is a set of all point within the circle $x^2 + y^2 = 16$ and outside the circle $x^2 + y^2 = 4$ excluding the points on both two cicles.

(ii) Given $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$

Using Sandwich theorem
$-1 \le \sin(xy) \le 1$
Multiply through by $\frac{1}{x^2+y^2}$, we get

$$-\frac{1}{x^2+y^2} \le \frac{\sin(xy)}{x^2+y^2} \le \frac{1}{x^2+y^2}$$

Consider, $\lim_{(x,y)\to(0,0)} -\frac{1}{x^2+y^2} = -\infty$

Consider, $\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = \infty$

$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$ *does not exist*

because $\lim_{(x,y)\to(0,0)} -\frac{1}{x^2+y^2} \neq \lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2}$

and also both $\lim_{(x,y)\to(0,0)} -\frac{1}{x^2+y^2}$ *and* $\lim_{(x,y)\to(0,0)} -\frac{1}{x^2+y^2}$ *are undefined.*

(b) $f(x,y) = \frac{x^3-y^3}{x-y}$, if $x \neq y$

For f to be continuous, $\lim_{x\to y} f(y,y) = f(y,y)$

Consider, $x \neq y$

$$\lim_{x\to y} \frac{x^3 - y^3}{x - y}$$

Using long division,

$$\begin{array}{r l} & x^2 + xy + y^2 \\ x - y & \sqrt{x^3 - y^3} \\ & \underline{-(x^3 - x^2y)} \\ & x^2y - y^3 \\ & \underline{-(x^2y - xy^2)} \\ & xy^2 - y^3 \\ & \underline{-(xy^2 - y^3)} \\ & 0 \end{array}$$

$$\Rightarrow \; x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

$$\begin{aligned} \Rightarrow \; \lim_{x\to y} \frac{x^3-y^3}{x-y} &= \lim_{x\to y} \frac{(x-y)(x^2+xy+y^2)}{x-y} \\ &= \lim_{x\to y} x^2 + xy + y^2 \\ &= y^2 + y \cdot y + y^2 \\ &= y^2 + y^2 + y^2 \\ &= 3y^2 \end{aligned}$$

Consider, $x = y$

$$\begin{aligned} f(y,y) &= y^2 + y \cdot y + y^2 \\ &= y^2 + y^2 + y^2 \\ &= 3y^2 \end{aligned}$$

Thus, the function is continuous at the point $x = y$

We redefined the function as

$$f(x,y) = \begin{cases} \frac{x^3 - y^3}{x - y} & if\ x \neq y \\ 3x^2 & if\ x = y \end{cases}$$

QUIZ 2

1. (a) Given that $U = \sqrt{x^2 + y^2}$; where $x = re^s$ and $y = re^{-s}$. Find $\frac{\partial U}{\partial r}$ and $\frac{\partial U}{\partial s}$.

(b) Let $f(x, y) = \frac{5}{x^2+y^2}$, find the linear approximation to the function at the point $(-1,2)$ and use it to approximate $f(-1.05, 2.1)$

(c) Determine whether or not the function

$$f(u, v) = \frac{u^3 + u^2v + uv^2 + v^3}{u^2 - v^2}$$

is homogeneous. If it is homogeneous, then find the degree of homogeneity of the function.

(d) Let

$$f(x, y) = \begin{cases} \dfrac{x^2 - xy}{x + y} & if\ (x, y) \neq (0,0) \\ 0 & if\ (x, y) = (0,0) \end{cases}$$

Find

i. $f_1(x, y)$, if $(x, y) \neq (0,0)$

ii. $f_1(0,0)$ and $f_2(0,0)$. [Hint: use limit definition for partial derivatives].

SOLUTION

1. (a) Let $u = f = \sqrt{x^2 + y^2} = (x^2 + y^2)^{\frac{1}{2}}$

$x = re^s$

$y = re^{-s}$

$\frac{\partial x}{\partial r} = e^s \qquad \frac{\partial x}{\partial s} = re^s$

$\frac{\partial y}{\partial r} = e^{-s} \qquad \frac{\partial y}{\partial s} = -re^{-s}$

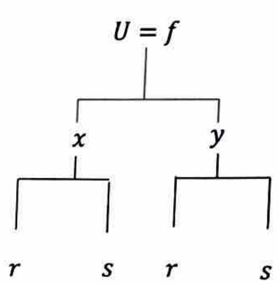
$f = (x^2 + y^2)^{\frac{1}{2}}$

$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}} \cdot 2x$

$= \frac{2x}{2(x^2+y^2)^{-\frac{1}{2}}} = \frac{x}{\sqrt{x^2+y^2}}$

$\frac{\partial f}{\partial y} = \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}} \cdot 2y$

$= \frac{2y}{2(x^2+y^2)^{-\frac{1}{2}}} = \frac{y}{\sqrt{x^2+y^2}}$



$$\frac{\partial U}{\partial r} = \frac{\partial f}{\partial r}$$
$$= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$$
$$= \frac{x}{\sqrt{x^2+y^2}}\cdot e^s + \frac{y}{\sqrt{x^2+y^2}}\cdot e^{-s}$$
$$= \frac{xe^s}{\sqrt{x^2+y^2}} + \frac{ye^{-s}}{\sqrt{x^2+y^2}}$$
$$= \frac{xe^s+ye^{-s}}{\sqrt{x^2+y^2}}$$

But $x = re^s$ and $y = re^{-s}$

$$\frac{\partial U}{\partial r} = \frac{re^s\cdot e^s+re^{-s}\cdot e^{-s}}{\sqrt{(re^s)^2+(re^{-s})^2}}$$
$$= \frac{re^{2s}+re^{-2s}}{\sqrt{r^2e^{2s}+r^2e^{-2s}}}$$
$$= \frac{r(e^{2s}+e^{-2s})}{\sqrt{r^2(e^{2s}+e^{-2s})}}$$
$$= \frac{r(e^{2s}+e^{-2s})}{r\sqrt{(e^{2s}+e^{-2s})}}$$
$$= \frac{e^{2s}+e^{-2s}}{\sqrt{(e^{2s}+e^{-2s})}}$$
$$= (e^{2s}+e^{-2s})^{1-\frac{1}{2}}$$
$$= (e^{2s}+e^{-2s})^{\frac{1}{2}}$$

$$\frac{\partial U}{\partial s} = \frac{\partial f}{\partial s}$$
$$= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$
$$= \frac{x}{\sqrt{x^2+y^2}}\cdot re^s + \frac{y}{\sqrt{x^2+y^2}}\cdot -re^{-s}$$
$$= \frac{xre^s}{\sqrt{x^2+y^2}} - \frac{yre^{-s}}{\sqrt{x^2+y^2}}$$
$$= \frac{xre^s-yre^{-s}}{\sqrt{x^2+y^2}}$$

But $x = re^s$ and $y = re^{-s}$

$$\frac{\partial U}{\partial r} = \frac{re^s\cdot re^s-re^{-s}\cdot re^{-s}}{\sqrt{(re^s)^2+(re^{-s})^2}}$$
$$= \frac{r^2e^{2s}-r^2e^{-2s}}{\sqrt{r^2e^{2s}+r^2e^{-2s}}}$$
$$= \frac{r^2(e^{2s}-e^{-2s})}{\sqrt{r^2(e^{2s}+e^{-2s})}}$$
$$= \frac{r^2(e^{2s}-e^{-2s})}{r\sqrt{(e^{2s}+e^{-2s})}}$$
$$= \frac{r(e^{2s}-e^{-2s})}{\sqrt{(e^{2s}+e^{-2s})}}$$

(b) Given $f(x,y) = \frac{5}{x^2+y^2}$ and $(-1,2)$

$$L(x,y) = f(a,b) + f_1(a,b)(x-a) + f_2(a,b)(y-b)$$
$$= f(-1,2) + f_1(-1,2)(x+1) + f_2(-1,2)(y-2)$$

Now, $f(-1,2) = \frac{5}{(-1)^2+(2)^2} = \frac{5}{1+4} = \frac{5}{5} = 1$

$$f_1(x,y) = \frac{\partial}{\partial x}\left(\frac{5}{x^2+y^2}\right)$$
$$= \frac{\partial}{\partial x}[5(x^2+y^2)^{-1}]$$
$$= -5(x^2+y^2)^{-2}\cdot 2x$$
$$= -\frac{10x}{(x^2+y^2)^2}$$

$$f_1(-1,2) = -\frac{10(-1)}{((-1)^2+(2)^2)^2} = \frac{10}{(1+4)^2} = \frac{10}{5^2} = \frac{2}{5}$$

$$f_2(x,y) = \frac{\partial}{\partial y}\left(\frac{5}{x^2+y^2}\right)$$
$$= \frac{\partial}{\partial y}[5(x^2+y^2)^{-1}]$$
$$= -5(x^2+y^2)^{-2}\cdot 2y$$
$$= -\frac{10y}{(x^2+y^2)^2}$$

$$f_2(-1,2) = -\frac{10(2)}{((-1)^2+(2)^2)^2} = -\frac{20}{(1+4)^2} = -\frac{20}{5^2} = -\frac{4}{5}$$

Thus, $L(x,y) = 1 + \frac{2}{5}(x+1) - \frac{4}{5}(y-2)$

$$= 1 + \frac{2}{5}x + \frac{2}{5} - \frac{4}{5}y + \frac{8}{5}$$

$$= \frac{2}{5}x - \frac{4}{5}y + 3$$

Thus, the linear approximation to $f(x,y)$ is $\frac{2}{5}x - \frac{4}{5}y + 3$.

Now, $L(-1.05,2.1) = \frac{2}{5}(-1.05) - \frac{4}{5}(2.1) + 3$

$$= -\frac{21}{50} - \frac{42}{25} + 3$$

$$= \frac{9}{10} = 0.9$$

Hence, the approximate of $f(-1.05,2.1)$ is 0.9

(c) Given $f(u,v) = \frac{u^3+u^2v+uv^2+v^3}{u^2-v^2}$

$$f(tu,tv) = \frac{(tu)^3+(tu)^2v+(tu)(tv)^2+(tv)^3}{(tu)^2-(tv)^2}$$

$$= \frac{t^3u^3+t^3u^2v+t^3uv^2+t^3v^3}{t^2u^2-t^2v^2}$$

$$= \frac{t^3(u^3+u^2v+uv^2+v^3)}{t^2(u^2-v^2)}$$

$$= \frac{t^3}{t^2}\left(\frac{u^3+u^2v+uv^2+v^3}{u^2-v^2}\right)$$

$$= t^1\big(f(u,v)\big)$$

Thus, $f(u,v)$ is a homogeneous function and hence, the degree of homogeneity of the function is $k = 1$

(d) Given $f(x,y) = \begin{cases} \frac{x^2-xy}{x+y} & if\ (x,y) \neq (0,0) \\ 0 & if\ (x,y) = (0,0) \end{cases}$

i. $f_1(x,y) = \frac{\partial}{\partial x} f(x,y)$

$$= \frac{\partial}{\partial x}\left(\frac{x^2-xy}{x+y}\right)$$

$$= \frac{(x+y)(2x-y)-(x^2-xy)(1)}{(x+y)^2}$$

$$= \frac{2x^2-xy+2xy-y^2-x^2+xy}{(x+y)^2}$$

$$= \frac{x^2+2xy-y^2}{(x+y)^2}$$

ii. $f_1(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$

$$f_1(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$$

$$f_1(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

But $f(x,y)\frac{x^2-xy}{x+y}$

$f(0,0) = 0$

$$f(h,0) = \frac{h^2-h(0)}{h+0} = \frac{h^2}{h} = h$$

$$f_1(0,0) = \lim_{h\to 0} \frac{h-0}{h}$$

$$= \lim_{h\to 0} \frac{h}{h}$$

$$= \lim_{h\to 0} 1$$

$$= 1$$

Thus, $f_1(x,y)$ at the point $(0,0)$ exist.

$$f_{12}(x,y) = \lim_{k\to 0} \frac{f_1(x,y+k)-f_1(x,y)}{k}$$

$$f_{12}(0,0) = \lim_{k\to 0} \frac{f_1(0,0+k)-f_1(0,0)}{k}$$

$$f_{12}(0,0) = \lim_{k\to 0} \frac{f_1(0,k)-f_1(0,0)}{k}$$

But $f_1(x,y) = \frac{x^2+2xy-y^2}{(x+y)^2}$

$f_1(0,0) = 1$

$$f_1(0,k) = \frac{0^2+2(0)(k)-k^2}{(0+k)^2} = \frac{-k^2}{k^2} = -1$$

$$f_{12}(0,0) = \lim_{k\to 0} \frac{-1-1}{k}$$

$$= \lim_{k\to 0} -\frac{2}{k}$$

$$= -\infty$$

Thus, $f_{12}(x,y)$ at the point $(0,0)$ does not exist.

EXAM –2021/22 (SECTION A)

1. If given $\varepsilon > 0, \exists\, \delta_\varepsilon > 0$ such that $|f(x,y) - L| < \varepsilon$ whenever (x,y) is in the domain of f and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then the function is
   A. Continuous at (a,b)
   B. Has a limit at the (a,b)
   C. uniformly continuous
   D. absolutely continuous at (a,b)

2. If given $\varepsilon > 0, \exists\, \delta_{\varepsilon,(a,b)} > 0$ such that $|f(x,y) - L| < \varepsilon$ whenever (x,y) is in the domain of f and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then the function is
   A. Continuous at (a,b)
   B. Continuous at (x,y)
   C. uniformly continuous
   D. absolutely continuous

3. What is the new limits of integration for the double integral
$$I = \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} f(x,y)\, dydx$$
   If the order of integration is reversed from $dydx$ to $dxdy$
   A. $I = \int_{y=x^2}^{y=x} \int_{x=0}^{x=1} f(x,y)\, dxdy$
   B. $I = \int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} f(x,y)\, dxdy$
   C. $I = \int_{y=0}^{y=1} \int_{x=\sqrt{y}}^{x=y} f(x,y)\, dxdy$
   D. $I = \int_{y=0}^{y=1} \int_{x=y}^{x=y^2} f(x,y)\, dxdy$

4. How can the function
$$f(x,y) = \begin{cases} x^2 + 2y & if\ (x,y) \neq (1,2) \\ 0 & if\ (x,y) = (1,2) \end{cases}$$
   be re-defined at $(1,2)$ so that f is continuous at all points in the xy-plane.
   A. $f(1,2) = 2$ B. $f(1,2) = 3$ C. $f(1,2) = 4$ D. $f(1,2) = 5$

5. A harmonic function of two variables satisfies
   A. Poisson equation
   B. Bernouli equation
   C. Laplace equation
   D. Heat equation

6. Find the linear approximation to the function
$$f(x,y,z) = xy + yz + zx$$
   at the point $(1,1,1)$.
   A. $L(1,1,1) = 2x - 2y + 2z - 3$
   B. $L(1,1,1) = 2x + 2y - 2z - 3$
   C. $L(1,1,1) = 2x + 2y + 2z + 3$
   D. $L(1,1,1) = 2x + 2y + 2z - 3$

7. What is the degree of homogeneity of
$$f(x,y,z) = \frac{\sqrt{x} + \sqrt{y} + \sqrt{z}}{x + y + z}.$$
   A. $\frac{1}{2}$ B. 1 C. $-\frac{1}{2}$ D. -1

8. Find $\frac{\partial}{\partial y} f(y^2, x^2)$

A. $2xyf_2(y^2, x^2)$ B. $2yf_1(y^2, x^2)$ C. $2xf_1(y^2, x^2)$ D. $2xf_2(y^2, x^2)$

9. In which region is the function $f(x, y) = \sqrt{1 - x^2 - y^2}$ continuous?
   A. f is continuous in the closed circle $x^2 + y^2 \leq 1$
   B. f is continuous in the closed circle $x^2 + y^2 < 1$
   C. f is continuous in the closed circle $x^2 - y^2 \leq 1$
   D. f is continuous in the closed circle $x^2 - y^2 < 1$

10. How can the function

$$f(x, y) = \frac{x^3 - y^3}{x - y} \quad if\ x \neq y$$

be redefined along the line $x = y$ so that the resulting function is continuous on the whole xy-plane.
   A. $f(x, y) = x^2 + y^2 - xy$
   B. $f(x, y) = x^2 + y^2 + xy$
   C. $f(x, y) = x^2 - y^2 + xy$
   D. $f(x, y) = x^2 - y^2 - xy$

11. If

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & if \quad (x, y) \neq (0,0) \\ 0 & if \ (x, y) = (0,0) \end{cases}$$

then $f_{yx}(0,0)$ is
   A. $f_{yx}(0,0) = \lim_{h \to 0} \frac{f_y(h,0) + f_y(0,0)}{h}$
   B. $f_{yx}(0,0) = \lim_{h \to 0} \frac{f_y(0,h) - f(0,0)}{h}$
   C. $f_{yx}(0,0) = \lim_{h \to 0} \frac{f(0,h) - f_y(0,0)}{h}$
   D. $f_{yx}(0,0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$

12. The range of $f(x, y, z) = \sqrt{16 - x^2 - y^2 - z^2}$ is
   A. $\{R_f : 0 \leq f(x, y, z) \leq 4\}$
   B. $\{R_f : 0 \leq f(x, y, z) \leq 2\sqrt{2}\}$
   C. $\{R_f : 0 \leq f(x, y, z) \leq 2\}$
   D. $\{R_f : 0 \leq f(x, y, z) \leq 3\sqrt{2}\}$

13. A point (x_0, y_0) is called a relative maximum point of $f(x, y)$ in the domain of f if,
   A. $f_{xx}f_{yy} + f_{xy}^2|_{(x_0,y_0)} > 0, \ f_{xx} < 0$
   B. $f_x f_y - f_{xy}^2|_{(x_0,y_0)} < 0, \ f_{xx} > 0$
   C. $f_{xx}f_{yy} - f_{xy}^2|_{(x_0,y_0)} > 0, \ f_{xx} < 0$
   D. $f_{xx}f_{yy} - f_{xy}|_{(x_0,y_0)} < 0, \ f_{xx} < 0$

14. Taylor's Theorem of the Mean states that:
   A. $f(x, y) = \sum_{m=0}^{n} \frac{1}{m!}\left(h\frac{\partial}{\partial x} - k\frac{\partial}{\partial y}\right)^m f(x_0, y_0) + R_n$
   where $R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0 + \theta h, y_0 + \theta k), \quad 0 < \theta < 1$
   B. $f(x, y) = \sum_{m=0}^{n} \frac{1}{m!}\left(h\frac{\partial}{\partial x} - k\frac{\partial}{\partial y}\right)^m f(x_0, y_0) + R_n$
   where $R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} - k\frac{\partial}{\partial y}\right)^n f(x_0 + \theta h, y_0 + \theta k), \quad 0 < \theta < 1$

C. $f(x,y) = \sum_{m=0}^{n} \frac{1}{m!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{m+1} f(x_0, y_0) + R_n$
where $R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0 + \theta h, y_0 + \theta k), \quad 0 < \theta < 1$

D. $f(x,y) = \sum_{m=0}^{n} \frac{1}{m!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{m} f(x_0, y_0) + R_n$
where $R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0 + \theta h, y_0 + \theta k), \quad 0 < \theta < 1$

15. Evaluate

$$\iiint_B xyz\, dV$$

if B is the rectangle box: $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.

A. $\frac{1}{8}$ B. $\frac{1}{6}$ C. $\frac{1}{4}$ D. $\frac{1}{2}$

16. Find the limits of integration for the area between the parabolas: $y = x^2$ and $y^2 = x$ if the integration is done first parallel to the y-axis followed by integration parallel to the x-axis.

A. $\int_{y=x^2}^{y=\sqrt{x}} \int_{x=0}^{x=1} dA$ C. $I = \int_{y=\sqrt{x}}^{y=x} \int_{x=0}^{x=1} dA$
B. $\int_{x=0}^{x=1} \int_{y=y}^{y=\sqrt{y}} dA$ D. $\int_{x=0}^{y=1} \int_{y=x^2}^{y=\sqrt{x}} dA$

17. Find the critical points of $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

A. $(\pm 1, \pm 3)$ B. $(\pm 1, \pm 2)$ C. $(\pm 1, 2)$ D. $(-1, \pm 2)$

18. If $x = u - v + w$, $y = u^2 - v^2$ and $z = u^3 + v$, evaluate the Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

A. $2u + 6u^2 v$ C. $2u - 6u^2 v$
B. $2u + 6uv$ D. $2u^2 + 6u^2 v$

19. Re write the equation $z = x^2 + y^2$ in spherical coordinate system.

A. $r^2 = -\rho \sin\phi$ C. $r^2 = -\rho\cos\phi$
B. $r^2 = \rho\sin\phi$ D. $r^2 = \rho\cos\phi$

20. Evaluate $\lim_{(x,y)\to(0,0)} \sin\left(\frac{xy}{x^2+y^2}\right)$.

A. 0 B. 1 C. 2 D. limit does not exist

21. Find the domain of $f(x,y) = \sin^{-1}(x + y - 1)$.

A. $-\frac{\pi}{2} \le x + y - 1 \le \frac{\pi}{2}$ C. $-1 \le x + y - 1 \le 1$
B. $-1 < x + y - 1 < 1$ D. $-\pi \le x + y - 1 \le \pi$

22. The domain of $f(x,y) = \ln(9 - x^2 - 9y^2)$.
   A. $D_f = \left\{(x,y) | \frac{x^2}{9} - y^2 < 1\right\}$
   B. $D_f = \left\{(x,y) | -\frac{x^2}{9} + y^2 < 1\right\}$
   C. $D_f = \left\{(x,y) | \frac{x^2}{9} + y^2 < 1\right\}$
   D. $D_f = \left\{(x,y) | -\frac{x^2}{9} - y^2 < 1\right\}$

23. Describe the set of points represented by domain of $f(x,y) = \ln(9 - x^2 - 9y^2)$.
   A. Set of points in an Ellipse excluding points on the boundary
   B. Set of points in an Ellipse including points on the boundary
   C. Set of points in an Circle excluding points on the boundary
   D. Set of points in an Circle including points on the boundary

24. Find $f(3,2)$ if $f(x,y) = x\ln(y^2 - x)$.
   A. 3 B. 2 C. 1 D. 0

25. Find $\lim_{(x,y)\to(0,0)} \frac{x\sin(x^2+y^2)}{x^2+y^2}$
   A. 0 B. 1 C. 2 D. 3

26. What should be $f(1,2)$ if the function

$$f(x,y) = \begin{cases} 3xy & if \quad (x,y) \neq (1,2) \\ 0 & if \ (x,y) = (1,2) \end{cases}$$

is to be continuous at (1,2).
   A. 0 B. 2 C. 4 D. 6

27. Determine the set of points for which $h(x,y) = \exp\left(\frac{x}{y}\right)$ is continuous
   A. h is continuous on the set $\{(x,y) : x \neq 0\}$
   B. h is continuous on the set $\{(x,y) : y = 0\}$
   C. h is continuous on the set $\{(x,y) : x, y \neq 0\}$
   D. h is continuous on the set $\{(x,y) : y \neq 0\}$

28. The partial derivative of $f(x,y,z)$ with respect of x is defined as
   A. $f_1(x,y,z) = \lim_{h_1\to 0} \frac{f(x-h_1,y,z)-f(x,y,z)}{h_1}$
   B. $f_1(x,y,z) = \lim_{h_1\to 0} \frac{f(x+h_1,y,z)-f(x,y,z)}{h_1}$
   C. $f_1(x,y,z) = \lim_{h_1\to 0} \frac{f(x,y+h_1,z)-f(x,y,z)}{h_1}$
   D. $f_1(x,y,z) = \lim_{h_1\to 0} \frac{f(x+h_1,y,z)+f(x,y,z)}{h_1}$

29. If

$$f(x,y) = \begin{cases} (x^3 + y)\sin\frac{1}{x^2+y^2} & if \quad (x,y) \neq (0,0) \\ 0 & if \ (x,y) = (0,0) \end{cases}$$

then $f_2(0,0)$ is
   A. $\lim_{k\to 0} \sin\left(\frac{1}{k}\right)$ B. $\lim_{k\to 0} \sin(k)$ C. $\lim_{k\to 0} \sin\left(\frac{1}{k^2}\right)$ D. $\lim_{k\to 0} \sin\left(-\frac{1}{k^2}\right)$

30. Find $\frac{\partial z}{\partial x}$ if $x^2z^2 + u\sin xz = 2$

A. $\frac{\partial z}{\partial x} = \frac{(2xz^2+uz\cos xz)}{2x^2z+ux\cos xz}$

B. $\frac{\partial z}{\partial x} = -\frac{(2xz^2+uz\cos xz)}{2x^2z+ux\cos xz}$

C. $\frac{\partial z}{\partial x} = -\frac{(2xz^2-uz\cos xz)}{2x^2z+ux\cos xz}$

D. $\frac{\partial z}{\partial x} = -\frac{(2xz^2+uz\cos xz)}{2x^2z-ux\cos xz}$

31. Find $f_{xy}(x,y)$ if $f(x,y) = \sin(x^2y)$.

A. $-2x\cos(x^2y) - 2x^3y\sin(x^2y)$

B. $2x\cos(x^2y) + 2x^3y\sin(x^2y)$

C. $2x\cos(x^2y) - 2x^3y\sin(x^2y)$

D. $-2x\cos(x^2y) + 2x^3y\sin(x^2y)$

32. If

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & if \quad (x,y) \neq (0,0) \\ 0 & if \ (x,y) = (0,0) \end{cases}$$

then $f_{xy}(0,0)$ is

A. $f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k)+f_x(0,0)}{k}$

B. $f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k)-f(0,0)}{k}$

C. $f_{xy}(0,0) = \lim_{k\to 0} \frac{f(0,k)-f_x(0,0)}{k}$

D. $f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k}$

33. Find the degree of homogeneity of $f(x,y) = x^2 + y$

A. not positively homogeneous B. 1 C. 2 D. 3

34. If z is a function of x and y with continuous first partial derivatives and if x and y depends on s and t, then $\frac{\partial z}{\partial s}$ is

A. $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$

B. $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$

C. $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{dz}{dy}\frac{\partial y}{\partial s}$

D. $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{dy}{ds}$

35. If $w = x^3y^3z^3$, then $\frac{\partial^2 w}{\partial x \partial y}$ is

A. $3x^3y^2z^3$ B. $3x^2y^2z^3$ C. $9x^2y^2z^2$ D. $9x^2y^2z^3$

36. If $F(x,y,z,u,v) = 0$, $G(x,y,z,u,v) = 0$, $H(x,y,z,u,v) = 0$, then $\left(\frac{\partial x}{\partial y}\right)_z$ is

A. $\left(\frac{\partial x}{\partial y}\right)_z = \frac{\frac{\partial(F,G,H)}{\partial(y,u,x)}}{\frac{\partial(F,G,H)}{\partial(x,u,v)}}$

B. $\left(\frac{\partial x}{\partial y}\right)_z = \frac{\frac{\partial(F,G,H)}{\partial(y,x,v)}}{\frac{\partial(F,G,H)}{\partial(x,u,v)}}$

C. $\left(\frac{\partial x}{\partial y}\right)_z = \frac{\frac{\partial(F,G,H)}{\partial(y,u,v)}}{\frac{\partial(F,G,H)}{\partial(z,u,v)}}$

D. $\left(\frac{\partial x}{\partial y}\right)_z = \frac{\frac{\partial(F,G,H)}{\partial(y,u,v)}}{\frac{\partial(F,G,H)}{\partial(x,u,v)}}$

37. Find $\frac{\partial z}{\partial x}$ if $z = f(x, y)$ is defined by the equation $2xz^3 - 3yz^2 + x^2y^2 + 4z = 0$.

A. $\frac{\partial z}{\partial x} = -\frac{(2z^3 - 2xy^2)}{6xz^2 - 6yz + 4}$
B. $\frac{\partial z}{\partial x} = -\frac{(2z^3 + 2xy^2)}{6xz^2 + 6yz + 4}$
C. $\frac{\partial z}{\partial x} = \frac{(2z^3 + 2xy^2)}{6xz^2 - 6yz + 4}$
D. $\frac{\partial z}{\partial x} = -\frac{(2z^3 + 2xy^2)}{6xz^2 - 6yz + 4}$

38. A point (x_0, y_0) is called a Saddle point of $f(x, y)$ in the domain of f if

A. $f_{xx}f_{yy} + f_{xy}^2|_{(x_0,y_0)} < 0$
B. $f_x f_y - f_{xy}^2|_{(x_0,y_0)} < 0$
C. $f_{xx}f_{yy} - f_{xy}^2|_{(x_0,y_0)} < 0$
D. $f_{xx}f_{yy} - f_{xy}|_{(x_0,y_0)} < 0$

39. Evaluate $\int_1^2 \int_0^2 (x^2 - 3y)\, dxdy$

A. $-\frac{19}{3}$ B. $\frac{19}{3}$ C. -3 D. $-\frac{1}{3}$

40. Evaluate $\int_0^2 \int_{y=x^2}^{y=2x} (x^3 + 4y)\, dxdy$

A. $-\frac{32}{3}$ B. $\frac{2}{3}$ C. 3 D. $\frac{32}{3}$

41. If R is the cube $0 \leq x, y, z \leq 1$, evaluate

$$\iiint_R (x^2 + y^2)\, dV$$

A. $-\frac{32}{3}$ B. $\frac{2}{3}$ C. 3 D. $\frac{32}{3}$

42. Evaluate $\int_1^4 \int_{-2}^0 \int_0^1 xyz\, dxdydz$

A. $-\frac{1}{2}$ B. $\frac{15}{2}$ C. $-\frac{15}{2}$ D. $-\frac{5}{2}$

43. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$ [Hint: use polar coordinates with $dA = rdrd\theta, 0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 1$]

A. $\frac{\pi}{5}$ B. $\frac{\pi}{3}$ C. $\frac{\pi}{4}$ D. $\frac{\pi}{2}$

44. The domain of $g(x, y) = \sqrt{\frac{xy}{x^2 + y^2}}$

A. $\{(x, y) : xy < 0,\ (x, y) = (0,0)\}$
B. $\{(x, y) : xy > 0,\ (x, y) \neq (0,0)\}$
C. $\{(x, y) : x > y,\ (x, y) \neq (0,0)\}$
D. $\{(x, y) : y > x,\ (x, y) = (0,0)\}$

45. Find the critical points if $f(x, y) = 2x^3 - 6xy + 3y^2$

A. $(0,0), (1,1)$ B. $(0,1), (1,-1)$ C. $(0,0), (1,0)$ D. $(1,0), (-1,1)$

46. The range of $f(x, y) = \sqrt{8 - x^2 - y^2}$ is

A. $\{R_f : 0 \leq f(x, y) \leq 3\}$
B. $\{R_f : 0 \leq f(x, y) \leq 2\sqrt{2}\}$
C. $\{R_f : 0 \leq f(x, y) \leq 2\}$
D. $\{R_f : 0 \leq f(x, y) \leq 3\sqrt{2}\}$

47. If given $\varepsilon > 0, \exists\ \delta_\varepsilon > 0$ such that $|f(x_1, y_1) - f(x_2, y_2)| < \varepsilon$ whenever (x_1, y_1) and $(2, y_2)$ is in the domain of f and $0 < \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} < \delta$ then the function is

A. Absolutely continuous
B. Continuous at (x_2, y_2)
C. uniformly continuous
D. Continuous at (x_1, y_1)

48. Evaluate $\lim_{(x,y)\to(4,1)} \frac{xy-4y^2}{\sqrt{x}-2\sqrt{y}}$

A. 2 B. 3 C. 4 D. 1

49. Evaluate $\lim_{(x,y,z)\to\left(2,\frac{\pi}{2},0\right)} \frac{x^2\sin(y)}{z^2+4}$

A. 1 B. 2 C. 3 D. 4

50. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

A. 0 B. limit does not exist C. 1 D. 2

51. Find $\frac{dz}{dt}$, where $z = f(x, y, t)$, $x = g(t)$ and $y = h(t)$. (Assume that f, g and h all have continuous derivatives)

A. $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial t}$
B. $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + \frac{dz}{dt}$
C. $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt} + \frac{\partial z}{\partial t}$
D. $\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial t}$

52. Let $u = x^2 + xy - y^2$ and $v = 2xy + y^2$, find $\left(\frac{\partial x}{\partial u}\right)_v$ at the point $(2, -1)$

A. $\frac{1}{6}$ B. $\frac{1}{7}$ C. $\frac{1}{8}$ D. $\frac{1}{9}$

53. Find $f_1(0, \pi)$ if $f(x, y) = [\cos(x + y)]\exp(xy)$

A. $-\pi$ B. π C. 0 D. -1

54. Find $\frac{\partial w}{\partial x}$ at the point $(2, 0, -1)$ if $w = \ln[1 + \exp(xyz)]$

A. 1 B. 3 C. 0 D. -3

55. If $w = f(x, y, z)$ where $x = g(y, z)$ and $y = h(z)$, state the appropriate version of the chain rule for $\left(\frac{\partial w}{\partial z}\right)_x$

A. $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial f}{\partial z}$
B. $\frac{\partial f}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial f}{\partial z}$
C. $\frac{\partial f}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial f}{\partial z}$
D. $\frac{\partial f}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial f}{\partial z}$

56. Given that $x = r\cos\theta$ and $x = r\sin\theta$, compute $\frac{\partial(x,y)}{\partial(r,\theta)}$

A. r^2 B. $r\sin\theta$ C. r D. $\frac{1}{r}$

57. Evaluate $\int_0^4 \int_0^4 \int_0^4 kz\,dxdydz$ where k is a constant.
A. $126k$ B. $127k$ C. $128k$ D. $129k$

58. Convert the cylindrical coordinate $\left(3, \frac{\pi}{3}, -4\right)$ to Cartesian coordinate.
A. $\left(\frac{1}{2}, \frac{3\sqrt{3}}{2}, -4\right)$ B. $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}, -4\right)$ C. $\left(2, \frac{3\sqrt{3}}{2}, -4\right)$ D. $\left(-2, \frac{3\sqrt{3}}{2}, -4\right)$

59. Convert the spherical coordinate $\left(4, \frac{\pi}{4}, \frac{\pi}{6}\right)$ to Cartesian coordinate.
A. $(\sqrt{2}, 2\sqrt{2}, 4\sqrt{3})$ B. $(2\sqrt{2}, 2\sqrt{2}, 4\sqrt{3})$ C. $(2\sqrt{2}, \sqrt{2}, 4\sqrt{3})$ D. $(2\sqrt{2}, 2\sqrt{2}, \sqrt{3})$

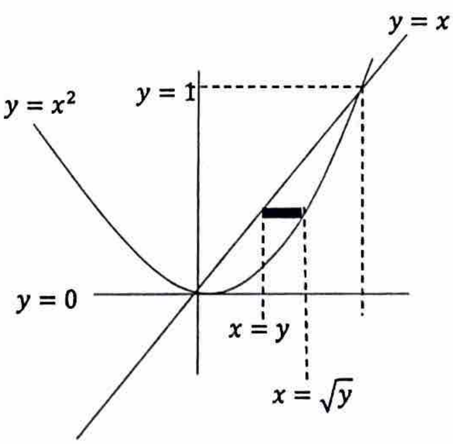
60. Find the relative maximum or minimum point of $f(x, y) = 2 - x^2 - xy - y^2$.
A. (0,0) is the rel. max. point C. (0,0) is the saddle point
B. (0,0) is the rel. min. point D. (0,0) is the point of inflection

SOLUTION

1. Has a limit at point (a, b)
Answer: B

2. Continuous at (a, b)
Answer: A

3.



From the diagram, by reversing the order from $dydx$ to $dxdy$, we have

$$I = \int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} f(x, y)\,dxdy$$

Answer: B

4. For continuity,
$\lim_{(x,y)\to(1,2)} f(x,y) = f(1,2)$
$f(1,2) = 1^2 + 2(2)$
$= 5$

$\lim_{(x,y)\to(1,2)} f(x,y) = \lim_{(x,y)\to(1,2)} f(x,y)$
$= \lim_{(x,y)\to(1,2)} x^2 + 2y$
$= 1^2 + 2(2) = 5$

Answer: D

5. Laplace equation
Answer: C

6. Given $f(x,y,z) = xy + yz + zx$
$L(x,y,z) = f(1,1,1) + f_x(1,1,1)(x-1) + f_y(1,1,1)(y-1) + f_z(1,1,1)(z-1)$

$f(1,1,1) = 1(1) + 1(1) + 1(1) = 3$

$f_x(x,y,z) = y + z \Rightarrow f_x(1,1,1) = 1 + 1 = 2$

$f_y(x,y,z) = x + z \Rightarrow f_y(1,1,1) = 1 + 1 = 2$

$f_z(x,y,z) = y + x \Rightarrow f_z(1,1,1) = 1 + 1 = 2$

$L(x,y,z) = 3 + 2(x-1) + 2(y-1) + 2(z-1)$
$= 3 + 2x - 2 + 2y - 2 + 2z - 2$
$= 2x + 2y + 2z - 3$

Answer: D

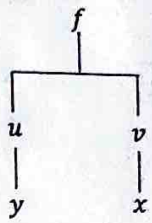
7. Given $f(x,y,z) = \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{x+y+z}$

$f(tx,ty,tz) = \frac{\sqrt{tx}+\sqrt{ty}+\sqrt{tz}}{tx+ty+tz}$
$= \frac{\sqrt{t}\sqrt{x}+\sqrt{t}\sqrt{y}+\sqrt{t}\sqrt{z}}{t(x+y+z)}$
$= \frac{\sqrt{t}(\sqrt{x}+\sqrt{y}+\sqrt{z})}{t(x+y+z)}$
$= \frac{t^{\frac{1}{2}}}{t}\left(\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{x+y+z}\right)$
$= t^{-\frac{1}{2}} f(x,y,z)$

Thus, $-\frac{1}{2}$ is the degree of homogeneity.
Answer: C

8. Let $u = y^2$ and $v = x^2$

$\frac{du}{dy} = 2y$ $\frac{du}{dx} = 2x$



$\frac{\partial}{\partial y} f(u, v) = \frac{\partial f}{\partial u} \cdot \frac{du}{dy}$

$\frac{\partial}{\partial y} f(u, v) = f_1(u, v) \cdot 2y$

$\frac{\partial}{\partial y} f(y^2, x^2) = 2y f_1(y^2, x^2)$

Answer: B

9. For $f(x, y) = \sqrt{1 - x^2 - y^2}$ to be defined,

$1 - x^2 - y^2 \geq 0$

$x^2 + y^2 \geq 1$

Thus, f is continuous in the closed circle $x^2 + y^2 \leq 1$

Answer: A

10. Given $f(x, y) = \frac{x^3 - y^3}{x - y}$

Using long division, we obtain $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Now, $f(x, y) = \frac{(x-y)(x^2+xy+y^2)}{x-y}$

$= x^2 + xy + y^2$

$= x^2 + y^2 + xy$

Answer: B

11. Given $f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & if \quad (x, y) \neq (0,0) \\ 0 & if \quad (x, y) = (0,0) \end{cases}$

Thus, $f_{yx}(0,0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$

Answer: D

12. Given $f(x, y, z) = \sqrt{16 - x^2 - y^2 - z^2}$ let $t = x^2 + y^2 + z^2$

$0 \leq 16 - t \leq 16$

$0 \leq 16 - (x^2 + y^2 + z^2) \leq 16$

$0 \leq 16 - x^2 - y^2 - z^2 \leq 16$

$\sqrt{0} \leq \sqrt{16 - x^2 - y^2 - z^2} \leq \sqrt{16}$

$0 \leq \sqrt{16 - x^2 - y^2 - z^2} \leq 4$

$0 \leq f(x, y, z) \leq 4$

Answer: A

13. The relative maximum point at (x_0, y_0) is $f_{xx}f_{yy} - f_{xy}^2|_{(x_0,y_0)} > 0, \quad f_{xx} < 0$

Answer: C

14. The Taylor's Theorem of the Mean states that

$$f(x,y) = \sum_{m=0}^{n} \frac{1}{m!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m f(x_0, y_0) + R_n$$

where $R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0 + \theta h, y_0 + \theta k), \quad 0 < \theta < 1$

Answer: D

15. Given $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.

$$\int_0^1\int_0^1\int_0^1 xyz\,dxdydz = \int_0^1\int_0^1\left[\frac{x^2}{2}yz\right]_{x=0}^{x=1} dydz$$

$$= \frac{1}{2}\int_0^1\int_0^1 yz\,dydz$$

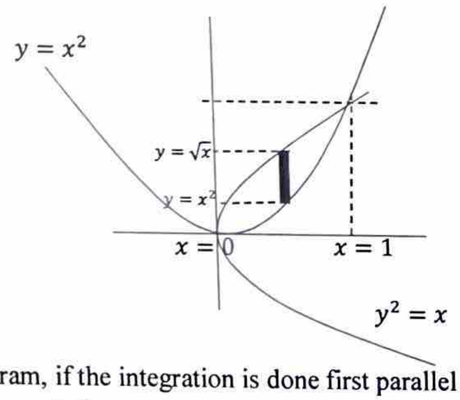
$$= \frac{1}{2}\int_0^1\left[\frac{y^2}{2}z\right]_{y=0}^{y=1} dz$$

$$= \frac{1}{4}\int_0^1 z\,dz$$

$$= \frac{1}{4}\left[\frac{z^2}{2}\right]_{y=0}^{y=1} = \frac{1}{8}$$

Answer: A

16. Given $y = x^2$ and $y^2 = x$



From the diagram, if the integration is done first parallel to the y-axis followed by integration parallel to the x-axis then

$$I = \int_{x=0}^{x=1}\int_{y=x^2}^{y=\sqrt{x}} dydx \int_{x=0}^{x=1}\int_{y=x^2}^{y=\sqrt{x}} dA$$

Answer: C

17. Given $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$f_x(x,y) = 3x^2 - 3$

$f_y(x,y) = 3y^2 - 12$

For critical points, $f_x(x,y) = f_y(x,y) = 0$

$3x^2 - 3 = 0 \qquad 3y^2 - 12 = 0$

$3x^2 = 3 \qquad 3y^2 = 12$

$x^2 = 1 \qquad y^2 = 4$

$x = \pm 1 \qquad y = \pm 2$

Thus, the critical points are $(\pm 1, \pm 2)$

Answer: B

18. Given $x = u - v + w, \ \ y = u^2 - v^2$ and $z = u^3 + v$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & 1 \\ 2u & -2v & 0 \\ 3u^2 & 1 & 0 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2u & -2v \\ 3u^2 & 1 \end{vmatrix} + 0 + 0$$

$$= 2u - (-6u^2v)$$

$$= 2u + 6u^2v$$

Answer: A

19. Given $z = x^2 + y^2$

In spherical coordinate system,

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\theta$

but $r^2 = x^2 + y^2 \ \ \Rightarrow \ \ r^2 = z$

Thus, $r^2 = \rho \cos\theta$

Answer: D

20. Given $\sin\left(\frac{xy}{x^2+y^2}\right)$

Using Sandwich theorem

$$-1 \le \sin\left(\frac{xy}{x^2+y^2}\right) \le 1$$

Consider, $\lim_{(x,y)\to(0,0)} -1 = -1$

Also, consider, $\lim_{(x,y)\to(0,0)} 1 = 1$

$\lim_{(x,y)\to(0,0)} \sin\left(\frac{xy}{x^2+y^2}\right)$ *does not exist because* $\lim_{(x,y)\to(0,0)} -1 \neq \lim_{(x,y)\to(0,0)} 1$

Answer: D

21. Given $f(x, y) = \sin^{-1}(x + y - 1)$
For real values of $f(x, y)$
$-1 \leq x + y - 1 \leq 1$
Answer: C

22. Given $f(x, y) = \ln(9 - x^2 - 9y^2)$
$D_f = \{(x, y) |\ 9 - x^2 - 9y^2 > 0\}$
$= \{(x, y) |\ x^2 + 9y^2 < 9\}$
$= \left\{(x, y) \middle|\ \frac{x^2}{9} + y^2 < 1\right\}$
Answer: C

23. Given $f(x, y) = \ln(9 - x^2 - 9y^2)$

$$9 - x^2 - 9y^2 > 0$$
$$x^2 + 9y^2 < 9$$
$$\frac{x^2}{9} + y^2 < 1$$
$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{1}\right)^2 < 1$$

Thus, domain is the set of points in an Ellipse excluding points on the boundary.
Answer: A

24. Given $f(x, y) = x\ln(y^2 - x)$

$$f(3,2) = 3\ln(2^2 - 3)$$
$$= 3\ln(4 - 3)$$
$$= 3\ln(1)$$
$$= 3(0)$$
$$= 0$$

Answer: D

25. Given $\lim\limits_{(x,y)\to(0,0)} \frac{x\sin(x^2+y^2)}{x^2+y^2}$

Using iterated limit,

$$\lim_{y\to 0}\left\{\lim_{x\to 0}\frac{x\sin(x^2 + y^2)}{x^2 + y^2}\right\} = \lim_{y\to 0}\left\{\frac{0 \cdot \sin(0^2 + y^2)}{0^2 + y^2}\right\}$$
$$= \lim_{y\to 0} 0 = 0$$

$$\lim_{x\to 0}\left\{\lim_{y\to 0}\frac{x\sin(x^2 + y^2)}{x^2 + y^2}\right\} = \lim_{x\to 0}\left\{\frac{x\sin(x^2 + 0^2)}{x^2 + 0^2}\right\}$$
$$= \lim_{x\to 0}\frac{x\sin x^x}{x^2}$$

$$= \lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{\sin x^x}{x^2}$$
$$= 0 \cdot 1$$
$$= 0$$

Confirming, using different path approach,
Consider, along $y = mx$

$$\lim_{x\to 0} \frac{x\sin(x^2 + (mx)^2)}{x^2 + (mx)^2} = \lim_{x\to 0} \frac{x\sin(x^2 + m^2y^2)}{x^2 + m^2y^2}$$
$$= \frac{0 \cdot \sin(0^2 + m^2y^2)}{0^2 + m^2y^2}$$
$$= 0$$

Answer: A

26. Given $f(x,y) = \begin{cases} 3xy & if \quad (x,y) \neq (1,2) \\ 0 & if \quad (x,y) = (1,2) \end{cases}$

For continuity,

$$\lim_{(x,y)\to(1,2)} f(x,y) = f(1,2)$$
$$f(1,2) = 3(1)(2)$$
$$= 6$$
$$\lim_{(x,y)\to(1,2)} f(x,y) = \lim_{(x,y)\to(1,2)} f(x,y)$$
$$= \lim_{(x,y)\to(1,2)} 3xy$$
$$= 3(1)(2)$$
$$= 6$$

Answer: D

27. Given $h(x,y) = \exp\left(\frac{x}{y}\right)$

For real value of $h(x,y)$,
$D_h = \{(x,y) : y \neq 0\}$
Thus, h is continuous on the set $\{(x,y) : y \neq 0\}$
Answer: D

28. The partial derivative of $f(x,y,z)$ with respect of x is defined as

$$f_1(x,y,z) = \lim_{h_1\to 0} \frac{f(x + h_1, y, z) - f(x,y,z)}{h_1}$$

Answer: B

29. Given $f(x,y) = \begin{cases} (x^3+y)\sin\frac{1}{x^2+y^2} & if \quad (x,y) \neq (0,0) \\ 0 & if \ (x,y) = (0,0) \end{cases}$

$$f_2(x,y) = \lim_{k\to 0}\frac{f(x,y+k)-f(x,y)}{k}$$
$$f_2(0,0) = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}$$
$$f(0,0) = 0$$
$$f(0,k) = (0^3+k)\sin\frac{1}{0^2+k^2}$$
$$= k\sin\left(\frac{1}{k^2}\right)$$
$$f_2(0,0) = \lim_{k\to 0}\frac{k\sin\left(\frac{1}{k^2}\right)-0}{k}$$
$$= \lim_{k\to 0}\frac{k\sin\left(\frac{1}{k^2}\right)}{k}$$
$$= \lim_{k\to 0}\sin\left(\frac{1}{k^2}\right)$$

Answer: C

30. Given $x^2z^2 + u\sin xz = 2 \implies x^2z^2 + u\sin xz - 2 = 0$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial}{\partial x}(x^2z^2+u\sin xz-2)}{\frac{\partial}{\partial z}(x^2z^2+u\sin xz-2)}$$
$$= -\frac{(2xz^2+uz\cos xz)}{2x^2z+ux\cos xz}$$

Answer: B

31. Given $f(x,y) = \sin(x^2y)$

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\sin(x^2y)\right)$$
$$= \frac{\partial}{\partial y}[2xy\cos(x^2y)]$$
$$= 2x\cos(x^2y) - 2xy\cdot x^2\sin(x^2y)$$
$$= 2x\cos(x^2y) - 2x^3y\sin(x^2y)$$

Answer: C

32. Given $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & if \quad (x,y) \neq (0,0) \\ 0 & if \ (x,y) = (0,0) \end{cases}$

Thus, $f_{xy}(0,0) = \lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k}$

Answer: D

33. Given $f(x, y) = x^2 + y$

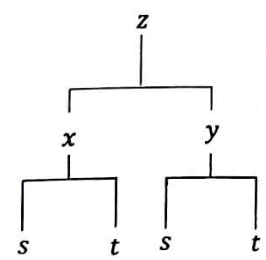
$$f(tx, ty) = (tx)^2 + ty$$
$$= t^2x^2 + ty$$
$$= t(tx^2 + y)$$

Since, $f(tx, ty) \neq t^k f(x, y)$

Thus, the function is not positively homogeneous.

Answer: A

34. Given that $z = f(x, y)$, $x = h(s, t)$ and $y = g(s, t)$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Answer: B

35. Given $w = x^3y^3z^3$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} w\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} x^3y^3z^3\right)$$
$$= \frac{\partial}{\partial x}(3x^3y^2z^3)$$
$$= 9x^2y^2z^3$$

Answer: D

36. Given $F(x, y, z, u, v) = 0$, $G(x, y, z, u, v) = 0$, $H(x, y, z, u, v) = 0$

$$\left(\frac{\partial x}{\partial y}\right)_z = \frac{\frac{\partial(F, G, H)}{\partial(y, u, v)}}{\frac{\partial(F, G, H)}{\partial(x, u, v)}}$$

Answer: D

37. Given $2xz^3 - 3yz^2 + x^2y^2 + 4z = 0$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial}{\partial x}(2xz^3 - 3yz^2 + x^2y^2 + 4z)}{\frac{\partial}{\partial z}(2xz^3 - 3yz^2 + x^2y^2 + 4z)}$$
$$= -\frac{(2z^3 - 0 + 2xy^2 + 0)}{6xz^2 - 6yz + 0 + 4}$$
$$= -\frac{(2z^3 + 2xy^2)}{6xz^2 - 6yz + 4}$$

Answer: B

38. The Saddle point at (x_0, y_0) is $f_{xx}f_{yy} - f_{xy}^2|_{(x_0,y_0)} < 0$

Answer: C

39. Given $\int_1^2 \int_0^2 (x^2 - 3y)\, dxdy$

$$\int_1^2 \int_0^2 (x^2 - 3y)\, dxdy = \int_1^2 \left[\frac{x^3}{3} - 3xy\right]_{x=0}^{x=2} dy$$
$$= \int_1^2 \left(\frac{8}{3} - 6y\right) dy$$
$$= \left[\frac{8}{3}y - 3y^2\right]_{y=1}^{y=2}$$
$$= \left(\frac{8}{3}(2) - 3(2)^2\right) - \left(\frac{8}{3}(1) - 3(1)^2\right)$$
$$= \left(\frac{16}{3} - 12\right) - \left(\frac{8}{3} - 3\right)$$
$$= -\frac{20}{3} + \frac{1}{3}$$
$$= -\frac{19}{3}$$

Answer: A

40. Given $\int_0^2 \int_{y=x^2}^{y=2x} (x^3 + 4y)\, dxdy$

$$\int_0^2 \int_{y=x^2}^{y=2x} (x^3 + 4y)\, dydx = \int_0^2 [x^3y + 2y^2]_{y=x^2}^{y=2x}\, dx$$
$$= \int_0^2 [(x^3(2x) + 2(2x)^2) - (x^3(x^2) + 2(x^2)^2)]\, dx$$

$$= \int_0^2 [(2x^4 + 8x^2) - (x^5 + 2x^4)]\, dx$$

$$= \int_0^2 [2x^4 + 8x^2 - x^5 - 2x^4]\, dx$$

$$= \int_0^2 [8x^2 - x^5]\, dx$$

$$= \left[\frac{8}{3}x^3 - \frac{x^6}{6}\right]_{x=0}^{x=2}$$

$$= \left(\frac{8}{3}(2)^3 - \frac{(2)^6}{6}\right) - \left(\frac{8}{3}(0)^2 - \frac{(0)^6}{6}\right)$$

$$= \left(\frac{64}{3} - \frac{64}{6}\right) - 0$$

$$= \frac{32}{3}$$

Answer: D

41. Given $0 \leq x, y, z \leq 1$,

$$\iiint_R (x^2 + y^2)\, dV = \int_0^1 \int_0^1 \int_0^1 (x^2 + y^2)\, dxdydz$$

$$= \int_0^1 \int_0^1 \left[\frac{x^3}{3} + xy^2\right]_{x=0}^{x=1} dydz$$

$$= \int_0^1 \int_0^1 \left(\frac{1}{3} + y^2\right) dydz$$

$$= \int_0^1 \left[\frac{1}{3}y + \frac{y^3}{3}\right]_{y=0}^{y=1} dz$$

$$= \int_0^1 \left(\frac{1}{3} + \frac{1}{3}\right) dz$$

$$= \int_0^1 \left(\frac{2}{3}\right) dz$$

$$= \left[\frac{2}{3}z\right]_{z=0}^{z=1}$$

$$= \frac{2}{3}$$

Answer: B

42. Given $\int_1^4 \int_{-2}^0 \int_0^1 xyz\,dxdydz$

$$\int_1^4 \int_{-2}^0 \int_0^1 xyz\,dxdydz = \int_1^4 \int_{-2}^0 \left[\frac{x^2}{2}\cdot yz\right]_{x=0}^{x=1} dydz$$

$$= \int_1^4 \int_{-2}^0 \frac{1}{2}\cdot yz\,dydz$$

$$= \frac{1}{2}\int_1^4 \left[\frac{y^2}{2}z\right]_{y=-2}^{y=0} dz$$

$$= \frac{1}{2}\int_1^4 (0-2z)\,dz$$

$$= \frac{1}{2}\int_1^4 (-2z)\,dz$$

$$= \frac{1}{2}\left[-2\cdot\frac{z^2}{2}\right]_{z=1}^{z=4}$$

$$= -\left[\frac{z^2}{2}\right]_{z=1}^{z=4}$$

$$= -\left[\frac{4^2}{2}-\frac{1^2}{2}\right]$$

$$= -\left(8-\frac{1}{2}\right)$$

$$= -\frac{15}{2}$$

Answer: C

43. Given $z = 1 - x^2 - y^2,\ \ z = 0,\ dA = rdrd\theta, \quad 0 \le \theta \le 2\pi\ and\ 0 \le r \le 1$

Using polar coordinates system,

$x = r\cos\theta \quad \Rightarrow x^2 = r^2\cos^2\theta$

$y = r\sin\theta \quad \Rightarrow y^2 = r^2\sin^2\theta$

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta)$

$x^2 + y^2 = r^2$

$z = 1 - x^2 - y^2$

$\quad = 1 - (x^2 + y^2)$

$\quad = 1 - r^2$

$$\int_0^{2\pi}\int_0^1 z\,dA = \int_0^{2\pi}\int_0^1 (1-r^2)\,rdrd\theta$$

$$= \int_0^{2\pi}\int_0^1 (r-r^3)\,drd\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{4}\right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{4}\right) d\theta$$

$$= \left[\frac{1}{4}\theta\right]_{\theta=0}^{\theta=2\pi}$$

$$= \frac{1}{4}(2\pi) = \frac{\pi}{2}$$

Answer: D

44. Given $g(x,y) = \sqrt{\frac{xy}{x^2+y^2}}$

For real values of $g(x,y)$,

$$\frac{xy}{x^2+y^2} \geq 0$$

But for the function to be defined, $x \neq 0,\ y \neq 0 \Rightarrow (x,y) \neq (0,0)$, then

$$\frac{xy}{x^2+y^2} > 0$$

$$(x^2+y^2) \cdot \frac{xy}{x^2+y^2} > (x^2+y^2) \cdot 0$$

$$xy > 0$$

Thus,

$$D_g = \{(x,y) : xy > 0, (x,y) \neq (0,0)\}$$

Answer: B

45. Given $f(x,y) = 2x^3 - 6xy + 3y^2$

$f_x(x,y) = 6x^2 - 6y$

$f_y(x,y) = -6x + 6y$

For critical points, $f_x(x,y) = f_y(x,y) = 0$

$6x^2 - 6y = 0$	$-6x + 6y = 0$
$6y = 6x^2$	$6x = 6y$
$y = x^2$	$x = y$

When $x = 0$, $y = 0$

When $x = 1$, $y = 1$

Thus, the critical points are (0,0) and (1,1)

Answer: B

46. Given $f(x,y) = \sqrt{8 - x^2 - y^2}$ let $z = x^2 + y^2$

$$0 \leq 8 - z \leq 8$$
$$0 \leq 8 - (x^2 + y^2) \leq 8$$
$$0 \leq 8 - x^2 - y^2 \leq 8$$
$$\sqrt{0} \leq \sqrt{8 - x^2 - y^2} \leq \sqrt{8}$$
$$0 \leq \sqrt{8 - x^2 - y^2 - z^2} \leq 2\sqrt{2}$$
$$0 \leq f(x,y) \leq 2\sqrt{2}$$

Answer: B

47. Uniformly continuous

Answer: C

48. Given $\lim_{(x,y)\to(4,1)} \frac{xy-4y^2}{\sqrt{x}-2\sqrt{y}}$

$$\lim_{(x,y)\to(4,1)} \frac{xy - 4y^2}{\sqrt{x} - 2\sqrt{y}} = \lim_{(x,y)\to(4,1)} \frac{xy - 4y^2}{\sqrt{x} - 2\sqrt{y}} \cdot \frac{\sqrt{x} + 2\sqrt{y}}{\sqrt{x} + 2\sqrt{y}}$$
$$= \lim_{(x,y)\to(4,1)} \frac{y(x - 4y)(\sqrt{x} + 2\sqrt{y})}{x + 2\sqrt{xy} - 2\sqrt{xy} - 4y}$$
$$= \lim_{(x,y)\to(4,1)} \frac{y(x - 4y)(\sqrt{x} + 2\sqrt{y})}{x - 4y}$$
$$= \lim_{(x,y)\to(4,1)} y(\sqrt{x} + 2\sqrt{y})$$
$$= 1(\sqrt{4} + 2\sqrt{1})$$
$$= 2 + 2$$
$$= 4$$

Answer: C

49. Given $\lim_{(x,y,z)\to\left(2,\frac{\pi}{2},0\right)} \frac{x^2\sin(y)}{z^2+4}$

$$\lim_{(x,y,z)\to\left(2,\frac{\pi}{2},0\right)} \frac{x^2 \sin(y)}{z^2 + 4} = \frac{2^2 \sin\left(\frac{\pi}{2}\right)}{0^2 + 4}$$
$$= \frac{2^2 \sin\left(\frac{\pi}{2}\right)}{0^2 + 4}$$
$$= \frac{4(1)}{4}$$
$$= 1$$

Answer: A

50. Given $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

Using iterated limit,

$$\lim_{y\to 0}\left\{\lim_{x\to 0}\frac{x^2}{x^2+y^2}\right\} = \lim_{y\to 0}\left\{\frac{0}{0^2+y^2}\right\}$$
$$= \lim_{y\to 0} 0$$
$$= 0$$

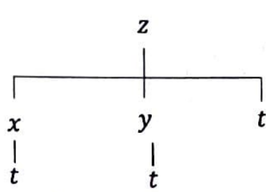
$$\lim_{x\to 0}\left\{\lim_{y\to 0}\frac{x^2}{x^2+y^2}\right\} = \lim_{x\to 0}\left\{\frac{x^2}{x^2+0^2}\right\}$$
$$= \lim_{x\to 0}\frac{x^2}{x^2}$$
$$= \lim_{x\to 0} 1$$
$$= 1$$

Since $\lim_{y\to 0}\left\{\lim_{x\to 0}\frac{x^2}{x^2+y^2}\right\} \neq \lim_{x\to 0}\left\{\lim_{y\to 0}\frac{x^2}{x^2+y^2}\right\}$, *then*

$\lim_{(x,y)\to(0,0)}\frac{x^2}{x^2+y^2}$ does not exist.

Answer: B

51. Given $z = f(x, y, t)$, $x = g(t)$ and $y = h(t)$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial t}$$

Answer: A

52. Given $u = x^2 + xy - y^2$ and $v = 2xy + y^2$

$$\frac{\partial}{\partial u}(u) = \frac{\partial}{\partial u}(x^2 + xy - y^2)$$
$$1 = 2x\frac{\partial x}{\partial u} + y\frac{\partial x}{\partial u} + x\frac{\partial y}{\partial u} - 2y\frac{\partial y}{\partial u}$$
$$1 = (2x + y)\frac{\partial x}{\partial u} + (x - 2y)\frac{\partial y}{\partial u} \text{ .... (1)}$$

$$\frac{\partial}{\partial u}(v) = \frac{\partial}{\partial u}(2xy + y^2)$$

$$0 = 2y\frac{\partial x}{\partial u} + 2(x+y)\frac{\partial y}{\partial u}$$

$$0 = y\frac{\partial x}{\partial u} + (x+y)\frac{\partial y}{\partial u} \dots\dots\dots (2)$$

Now, multiply (1) by $(x+y)$ and (2) by $(x-2y)$, we have

$$(x+y) = (x+y)(2x+y)\frac{\partial x}{\partial u} + (x+y)(x-2y)\frac{\partial y}{\partial u} \dots\dots (3)$$

$$0 = y(x-2y)\frac{\partial x}{\partial u} + (x+y)(x-2y)\frac{\partial y}{\partial u} \dots\dots\dots (4)$$

Subtracting (4) from (3), we obtain

$$(x+y) = (x+y)(2x+y)\frac{\partial x}{\partial u} - y(x-2y)\frac{\partial x}{\partial u}$$

$$(x+y) = (x+y)(2x+y)\frac{\partial x}{\partial u} - y(x-2y)\frac{\partial x}{\partial u}$$

$$(x+y) = [(x+y)(2x+y) - y(x-2y)]\frac{\partial x}{\partial u}$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{(x+y)}{(x+y)(2x+y) - y(x-2y)}$$

Evaluating at the point $(2, -1)$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{(2-1)}{\big(2+(-1)\big)\big(2(2)+(-1)\big) - (-1)\big(2-2(-1)\big)}$$

$$= \frac{1}{(2-1)(4-1)+(2+2)}$$

$$= \frac{1}{3+4}$$

$$= \frac{1}{7}$$

Answer: B

53. Given $f(x,y) = \cos(x+y)\,e^{xy}$

$$f_1(x,y) = ye^{xy}\cos(x+y) - e^{xy}\sin(x+y)$$

$$f_1(0,\pi) = \pi e^{(0)(\pi)}\cos(0+\pi) - e^{(0)(\pi)}\sin(0+\pi)$$

$$= \pi e^0\cos(\pi) - e^0\sin(\pi)$$

$$= \pi(1)(-1) - (1)(0)$$

$$= -\pi$$

Answer: A

54. Given $w = \ln[1 + \exp(xyz)]$

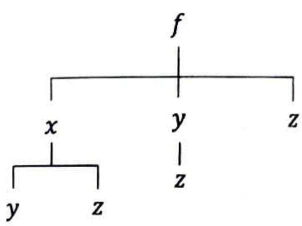
$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\ln[1 + \exp(xyz)]$$
$$= \frac{1}{1 + e^{xyz}} \cdot \frac{\partial}{\partial x}[1 + e^{xyz}]$$
$$= \frac{1}{1 + e^{xyz}} \cdot yze^{xyz}$$
$$= \frac{yze^{xyz}}{1 + e^{xyz}}$$

At the point $(2,0,-1)$

$$\frac{\partial w}{\partial x} = \frac{(0)(-1)e^{(2)(0)(-1)}}{1 + e^{(2)(0)(-1)}}$$
$$= \frac{0}{2}$$
$$= 0$$

Answer: C

55. Given $w = f(x, y, z)$ where $x = g(y, z)$ and $y = h(z)$



$$\left(\frac{\partial w}{\partial z}\right)_x = \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial f}{\partial z}$$

Answer: B

56. Given $x = r\cos\theta$ and $y = r\sin\theta$

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$
$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$
$$= r\cos^2\theta - (-r\sin^2\theta)$$
$$= r(\cos^2\theta + \sin^2\theta)$$
$$= r$$

Answer: A

57. Given $\int_0^4\int_0^4\int_0^4 kz\,dxdydz$

$$\int_0^4\int_0^4\int_0^4 kz\,dxdydz = k\int_0^4\int_0^4 [xz]_{x=0}^{x=4}\,dydz$$
$$= 4\int_0^4\int_0^4 4z\,dydz$$
$$= 4k\int_0^4 [yz]_{y=0}^{y=4}\,dz$$
$$= 4k\int_0^4 4z\,dz$$
$$= 16k\left[\frac{z^2}{2}\right]_{z=0}^{z=4}$$
$$= 16k\left[\frac{4^2}{2}-\frac{0^2}{2}\right]$$
$$= 16k(8-0)$$
$$= 128k$$

Answer: C

58. Given $\left(3,\frac{\pi}{3},-4\right) = (r,\theta,z)$

In the cylindrical coordinate system to Cartesian coordinate system (x,y,z)

$$x = r\cos\theta = 3\cos\left(\frac{\pi}{3}\right) = \frac{3}{2}$$
$$y = r\sin\theta = 3\sin\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$$
$$z = z = -4$$
$$(x,y,z) = \left(\frac{3}{2},\frac{3\sqrt{3}}{2},-4\right)$$

Answer: B

59. Given $\left(8,\frac{\pi}{4},\frac{\pi}{6}\right) = (\rho,\theta,\phi)$

In the spherical coordinate system to Cartesian system coordinate (x,y,z)

$$x = \rho\sin\phi\cos\theta$$
$$= 8\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right)$$
$$= 8\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= 2\sqrt{2}$$

$$y = \rho\sin\phi\sin\theta$$

$= 8\sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)$

$= 8\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$= 2\sqrt{2}$

$z = \rho\cos\phi$

$= 8\cos\left(\frac{\pi}{6}\right)$

$= 8\left(\frac{\sqrt{3}}{2}\right)$

$= 4\sqrt{3}$

$(x, y, z) = (2\sqrt{2}, 2\sqrt{2}, 4\sqrt{3})$

Answer: B

60. Given $f(x, y) = 2 - x^2 - xy - y^2$

$f_x(x, y) = -2x - y$

$f_y(x, y) = -x - 2y$

For critical points, $f_x(x, y) = f_y(x, y) = 0$

$-2x - y = 0$ $\qquad$ $-x - 2y = 0$

$2x = y$ $\qquad$ $x = 2y$

By solving, $2x = y$ and $x = 2y$ simultaneously, we obtain

$x = 0, y = 0$

Thus, the critical point is (0,0)

Checking the nature, using Jacobian,

$$\Delta J = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}_{(0,0)}$$

$$= \begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix}_{(0,0)}$$

$= 4 - 1$

$= 3 > 0$

Also,

$f_{xx}|_{(0,0)} = -2 < 0$

Since, $\Delta J > 0$ and $f_{xx} < 0$ at (0,0) then the critical point (0,0) is the relative maximum point.

Answer: A

2020/21 (QUIZ 1)

1. (a) Plot the following set in $\mathbb{R}^2$

$$\{(x,y) \in R^2 : x^2 + 2y^2 \le 2,\ 0 < x < y\}$$

(b) Let

$$f(x,y) = \frac{5x^2y^2}{x^2+y^2}$$

i. Find $\lim_{(x,y)\to(0,0)} f(x,y)$ using different path approach and hence;

ii. Verify using $\varepsilon - \delta$ definition of limit.

2. (a) Let

$$f(x,y) = \begin{cases} \frac{\cos y \sin x}{x} & if \quad (x,y) \neq (0,0) \\ \cos y & if \quad (x,y) = (0,0) \end{cases}$$

is f continuous at (0,0)? Justify.

(b) By using $\varepsilon - \delta$ definition of limit, prove that the function $f(x,y) = x^2 + y^2$ is continuous at point (1,1).

SOLUTION

1. (a) Given $\{(x,y) \in R^2 : x^2 + 2y^2 \le 2,\ 0 < x < y\}$

Consider, $x^2 + 2y^2 \le 2$

$$\frac{x^2}{2} + \frac{2y^2}{2} \le \frac{2}{2}$$

$$\frac{(x-0)^2}{(\sqrt{2})^2} + \frac{(y-0)^2}{1} \le 1$$

$$\left(\frac{x-0}{\sqrt{2}}\right)^2 + \left(\frac{y-0}{1}\right)^2 \le 1$$

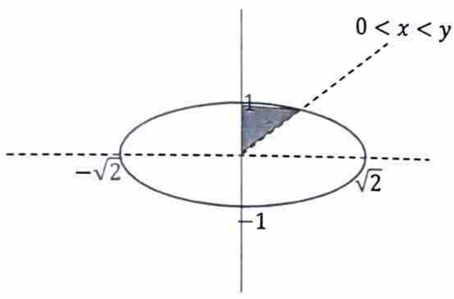
Thus, the domain is an ellipse with center (0,0) and major radius $\sqrt{2}$ and minor radius 1.

Also consider, $0 < x < y$

$$\Rightarrow x = 0, x = y,\ y = 0$$

Thus, the graph is a straight line starting from the origin

Sketch



Thus, the domain is the shaded portion.

(b) Given $f(x,y)=\frac{5x^2y^2}{x^2+y^2}$

i. Using different path approach

We consider $y=mx,\ y=x^2,\ x=0$

Along $y=mx$

$$f(x,mx)=\frac{5x^2(mx)^2}{x^2+(mx)^2}$$
$$=\frac{5x^2\cdot m^2\cdot x^2}{x^2+m^2\cdot x^2}$$
$$=\frac{5x^4m^2}{x^2(1+m^2)}$$
$$=\frac{5x^2m^2}{(1+m^2)}$$
$$\Rightarrow \lim_{x\to 0}\frac{5x^2m^2}{(1+m^2)}=\frac{5(0)^2m^2}{(1+m^2)}=0$$

Along $y=x^2$

$$f(x,x^2)=\frac{5x^2(x^2)^2}{x^2+(x^2)^2}$$
$$=\frac{5x^2\cdot x^4}{x^2+x^4}$$
$$=\frac{5x^6}{x^2(1+x^2)}$$
$$=\frac{5x^4}{(1+x^2)}$$
$$\Rightarrow \lim_{x\to 0}\frac{5x^4}{(1+x^2)}=\frac{5(0)^4}{(1+0^2)}=0$$

Also along $x=0$

$$f(0,y)=\frac{5(0)^2y^2}{(0)^2+y^2}$$
$$=\frac{0}{y^2}$$
$$=0$$
$$\Rightarrow \lim_{y\to 0}0=0$$

Since, different path approach have the same limit, we suspect that $\lim_{(x,y)\to(0,0)}f(x,y)=0$

ii. Verify using $\varepsilon - \delta$ definition of limit

Given $\varepsilon > 0, \exists\, \delta_\varepsilon > 0$ such that whenever $|x-0| < \delta$ and $|y-0| < \delta$ holds then $\left|\frac{5x^2y^2}{x^2+y^2} - 0\right| < \varepsilon$

$$\left|\frac{5x^2y^2}{x^2+y^2} - 0\right| = \left|\frac{5x^2y^2}{x^2+y^2}\right|$$
$$= \left|5x^2 \cdot \frac{y^2}{x^2+y^2}\right|$$
$$\leq 5|x^2| \cdot \frac{y^2}{x^2+y^2}$$
$$\leq 5|x^2| \cdot (1) \text{ since, } \frac{y^2}{x^2+y^2} < 1$$
$$= 5|x-0|^2$$
$$< 5\delta^2$$
$$\text{If } \delta \leq 1$$
$$\leq 5\delta$$
$$< \varepsilon \quad \text{if } \delta = \frac{\varepsilon}{5}$$

By choosing $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$ we can see that whenever $|x-0| < \delta$ and $|y-0| < \delta$ holds then $\left|\frac{5x^2y^2}{x^2+y^2} - 0\right| < \varepsilon$

Therefore, $\lim\limits_{(x,y)\to(0,0)} \frac{5x^2y^2}{x^2+y^2} = 0.$

2. (a) Given $f(x,y) = \begin{cases} \frac{\cos y \sin x}{x} & if \quad (x,y) \neq (0,0) \\ \cos y & if \quad (x,y) = (0,0) \end{cases}$

For f to be continuous,
$\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Thus, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

By definition, $f(x,y) = \cos y$ at $(x,y) = (0,0)$
$$f(0,0) = \cos 0$$
$$= 1$$

Thus, $f(0,0)$ exist

For $f(x,y) = \frac{\cos y \sin x}{x}$ at $(x,y) \neq (0,0)$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\cos y \sin x}{x}$$
$$= \lim_{y\to 0}\left\{\lim_{x\to 0}\frac{\cos y \sin x}{x}\right\}$$
$$= \lim_{y\to 0}\left\{\cos y \lim_{x\to 0}\frac{\sin x}{x}\right\}$$
$$= \lim_{y\to 0}\{\cos y\,(1)\}$$
$$= \lim_{y\to 0}\{\cos y\}$$
$$= \cos 0$$
$$= 1$$

Also, $\lim_{(x,y)\to(0,0)} f(x,y)$ exist

Hence, f is continuous at $(0,0)$ since $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$.

(b) Given $f(x,y) = x^2 + y^2$ at the point $(1,1)$
using $\varepsilon - \delta$ definition of limit,
Given $\varepsilon > 0, \exists\, \delta_{\varepsilon,(1,1)} > 0$ such that whenever $|x-1| < \delta$ and $|y-1| < \delta$ holds then $|x^2 + y^2 - f(1,1)| < \varepsilon$

$f(1,1) = 1^2 + 1^2 = 2$
Thus,

$$|x^2 + y^2 - f(1,1)| = |x^2 + y^2 - 2|$$
$$= |x^2 - 1 + y^2 - 1|$$
$$= |(x^2 - 1^2) + (y^2 - 1^2)|$$
$$= |(x+1)(x-1) + (y+1)(y-1)|$$
$$\leq |x+1||x-1| + |y+1||y-1|$$
$$< |x+1|\delta + |y+1|\delta$$
$$= |x-1+2|\delta + |y-1+2|\delta$$
$$= (|x-1|+2)\delta + (|y-1|+2)\delta$$
$$< (\delta+2)\delta + (\delta+2)\delta$$
$$< \delta^2 + 2\delta + \delta^2 + 2\delta$$

If $\delta \leq 1$

$$< \delta + 2\delta + \delta + 2\delta$$
$$= 6\delta$$
$$< \varepsilon \quad \text{if } \delta = \frac{\varepsilon}{6}$$

By choosing $\delta = \min\left\{1, \frac{\varepsilon}{6}\right\}$ we can see that whenever $|x-1| < \delta$ and $|y-1| < \delta$ holds then $|x^2 + y^2 - 2| < \varepsilon$

Therefore, the function $f(x,y) = x^2 + y^2$ is continuous at point $(1,1)$.

QUIZ 2

1. (a) Let

$$f(x,y) = \begin{cases} \dfrac{x^2 - xy}{x + y} & if\ (x,y) \neq (0,0) \\ 0 & if\ (x,y) = (0,0) \end{cases}$$

i. Find f_1 and f_{12}at the point $(x,y) \neq (0,0)$

ii. Evaluate $f_{12}(0,0)$ using the limit definition

(b) If $w = f(u,v)$ and $u = r\cos\theta$, $v = r\sin\theta$, show that

$$\left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$$

2. (a) i. Using suitable linearization, find an approximate value of the function $f(x,y) = \ln(x - 3y)$ at $(6.9, 2.06)$

ii. Find the degree of homogeneity of the function $f(x,y) = xy\tan\left(\frac{y}{x}\right)$.

SOLUTION

1. (a) Given $f(x,y) = \begin{cases} \frac{x^2-xy}{x+y} & if\ (x,y) \neq (0,0) \\ 0 & if\ (x,y) = (0,0) \end{cases}$

i. $f_1(x,y) = \frac{\partial}{\partial x} f(x,y)$

$$= \frac{\partial}{\partial x}\left(\frac{x^2-xy}{x+y}\right)$$

$$= \frac{(x+y)(2x-y)-(x^2-xy)(1)}{(x+y)^2}$$

$$= \frac{2x^2-xy+2xy-y^2-x^2+xy}{(x+y)^2}$$

$$= \frac{x^2+2xy-y^2}{(x+y)^2}$$

$$f_{12}(x,y) = \frac{\partial}{\partial y} f_1(x,y)$$

$$= \frac{\partial}{\partial y}\left(\frac{x^2+2xy-y^2}{(x+y)^2}\right)$$

$$= \frac{(x+y)^2(2x-2y)-(x^2+2xy-y^2)\cdot 2(x+y)}{(x+y)^4}$$

$$= \frac{(x+y)[(x+y)(2x-2y)-2(x^2+2xy-y^2)]}{(x+y)^4}$$

$$= \frac{2x^2-2xy+2xy-2y^2+2x^2-4xy+2y^2}{(x+y)^3}$$

$$= \frac{4x^2-4xy}{(x+y)^3}$$

$$= \frac{4x(x-y)}{(x+y)^3}$$

ii. $f_{12}(x,y) = \lim_{k\to 0} \frac{f_1(x,y+k)-f_1(x,y)}{k}$

$$f_{12}(0,0) = \lim_{k\to 0} \frac{f_1(0,0+k) - f_1(0,0)}{k}$$

$$f_{12}(0,0) = \lim_{k\to 0} \frac{f_1(0,k) - f_1(0,0)}{k}$$

But $f_1(x,y) = \frac{x^2+2xy-y^2}{(x+y)^2}$

$f_1(0,0) = 1$

$$f_1(0,k) = \frac{0^2 + 2(0)(k) - k^2}{(0+k)^2} = \frac{-k^2}{k^2} = -1$$

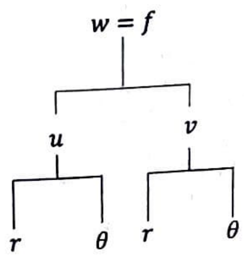
$$f_{12}(0,0) = \lim_{k\to 0} \frac{-1-1}{k}$$

$$= \lim_{k\to 0} -\frac{2}{k}$$

$$= -\infty$$

Thus, $f_{12}(x,y)$ at the point (0,0) does not exist.

(b) Given $w = f(u,v)$ and $u = r\cos\theta,\ v = r\sin\theta$



Consider,

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial r} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial r}$$

$$\frac{\partial w}{\partial r} = f_1 \cos\theta + f_2 \sin\theta$$

Squaring both sides, we have

$$\left(\frac{\partial w}{\partial r}\right)^2 = (f_1\cos\theta + f_2\sin\theta)^2$$

$$= f_1^2\cos^2\theta + 2f_1f_2\sin\theta\cos\theta + f_2^2\sin^2\theta$$

$$= f_1^2\cos^2\theta + f_2^2\sin^2\theta + 2f_1f_2\sin\theta\cos\theta \ \ldots\ldots\ldots (1)$$

Also,

$$\frac{\partial w}{\partial \theta} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial \theta} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial \theta}$$

$$\frac{\partial w}{\partial \theta} = f_1(-r\sin\theta) + f_2(r\cos\theta)$$

$$\frac{\partial w}{\partial \theta} = -f_1 r\sin\theta + f_2 r\cos\theta$$

Squaring both sides, we have

$$\left(\frac{\partial w}{\partial \theta}\right)^2 = (-f_1 r\sin\theta + f_2 r\cos\theta)^2$$

$$= f_1^2 r^2 \sin^2\theta - 2f_1 f_2 r^2 \sin\theta\cos\theta + f_2^2 r^2 \cos^2\theta$$

$$= f_1^2 r^2 \sin^2\theta + f_2^2 r^2 \cos^2\theta - 2f_1 f_2 r^2 \sin\theta\cos\theta \ldots\ldots.. (2)$$

Multiplying (2) by $\frac{1}{r^2}$

$$\frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \frac{1}{r^2}[f_1^2 r^2 \sin^2\theta + f_2^2 r^2 \cos^2\theta - 2f_1 f_2 r^2 \sin\theta\cos\theta]$$

$$= f_1^2 \sin^2\theta + f_2^2 \cos^2\theta - 2f_1 f_2 \sin\theta\cos\theta \ldots\ldots.. (3)$$

Adding (1) and (3), we obtain

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = f_1^2\cos^2\theta + f_2^2\sin^2\theta + 2f_1 f_2\sin\theta\cos\theta +$$

$$f_1^2\sin^2\theta + f_2^2\cos^2\theta - 2f_1 f_2\sin\theta\cos\theta$$

$$= f_1^2\cos^2\theta + f_2^2\sin^2\theta + f_1^2\sin^2\theta + f_2^2\cos^2\theta$$

$$= f_1^2\cos^2\theta + f_1^2\sin^2\theta + f_2^2\sin^2\theta + f_2^2\cos^2\theta$$

$$= f_1^2(\cos^2\theta + \sin^2\theta) + f_2^2(\sin^2\theta + \cos^2\theta)$$

But $\sin^2\theta + \cos^2\theta = 1$

$$= f_1^2 + f_2^2$$

$$= \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 \qquad but\ \ f = w$$

Hence, $\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2$

2. (a) Given $f(x,y) = \ln(x-3y)$ at $(6.9, 2.06)$
The nearest point is $(7,2)$

i. $$L(x,y) = f(a,b) + f_1(a,b)(x-a) + f_2(a,b)(y-b)$$
$$= f(7,2) + f_1(7,2)(x-7) + f_2(7,2)(y-2)$$

Now, $f(7,2) = \ln(7-3(2)) = \ln(7-6) = \ln(1) = 0$

$$f_1(x,y) = \frac{\partial}{\partial x}(\ln(x-3y))$$
$$= \frac{1}{x-3y} \cdot (1)$$
$$= \frac{1}{x-3y}$$
$$f_1(7,2) = \frac{1}{7-3(2)} = \frac{1}{7-6} = 1$$

$$f_2(x,y) = \frac{\partial}{\partial y}(\ln(x-3y))$$
$$= \frac{1}{x-3y} \cdot (-3)$$
$$= -\frac{3}{x-3y}$$
$$f_2(7,2) = -\frac{3}{7-3(2)} = -\frac{3}{7-6} = -3$$

Thus, $L(x,y) = 0 + 1(x-7) - 3(y-2)$
$$= x - 7 - 3y + 6$$
$$= x - 3y - 1$$
Thus, the linear approximation to $f(x,y)$ is $x - 3y - 1$

Now, $L(6.9, 2.06) = 6.9 - 3(2.06) - 1$
$$= 5.9 - 6.18$$
$$= -0.28$$

Hence, the approximate of $f(6.9, 2.06)$ is -2.28

ii. Given $f(x,y) = xy\tan\left(\frac{y}{x}\right)$

$$f(tx,ty) = (tx)(ty)\tan\left(\frac{(ty)}{(tx)}\right)$$
$$= t(xy)\tan\left(\frac{ty}{tx}\right)$$
$$= t\left[xy\tan\left(\frac{y}{x}\right)\right]$$
$$= t[f(x,y)]$$

Hence, the degree of homogeneity of the function $f(x,y)$ is 1

EXAMS (2020/21)

1. Find the domain of $f(x,y) = \sin^{-1}(x+y-1)$.
   A. $-\frac{\pi}{4} \le x+y-1 \le \frac{\pi}{4}$
   B. $-2 \le x+y-1 \le 2$
   C. $-1 \le x+y-1 \le 1$
   D. $-\frac{\pi}{2} \le x+y-1 \le \frac{\pi}{2}$

2. Evaluate $\lim_{(x,y)\to(4,1)} \frac{xy-4y^2}{\sqrt{x}-2\sqrt{y}}$
   A. 3 B. 4 C. 5 D. 2

3. Find $\lim_{(x,y)\to(e,1)} \frac{x}{y}$
   A. e B. 1 C. $\frac{1}{e}$ D. 2

4. Given
   $$F(x,y,z,u,v) = xe^y + uz - \cos v - 2, \qquad G(x,y,z,u,v) = u\cos y + x^2v - yz^2 - 1$$
   Evaluate $\frac{\partial(F,G)}{\partial(u,v)}$ at the point $(x,y,z,u,v) = (2,0,1,1,0)$.
   A. 1 B. 2 C. 3 D. 4

5. Determine the set of points at which the function $h(x,y) = \tan^{-1}(x+\sqrt{y})$ is continuous.
   A. $\{(x,y)\,|x \in R \text{ and } y \ge 0\}$
   B. $\{(x,y)\,|x > 0 \text{ and } y \ge 0\}$
   C. $\{(x,y)\,|x \in R \text{ and } y > 0\}$
   D. $\{(x,y)\,|x > 0 \text{ and } y > 0\}$

6. How can the function
   $$f(x,y) = \begin{cases} x^2 + 2y & if\ (x,y) \ne (1,2) \\ 0 & if\ (x,y) = (1,2) \end{cases}$$
   be re-defined at $(1,2)$ so that f is continuous at all points in the xy-plane.
   A. $f(1,2) = 4$ B. $f(1,2) = 5$ C. $f(1,2) = 6$ D. $f(1,2) = 7$

7. Find the degree of homogeneity of $f(x) = \ln x$.
   A. 2 B. There is no such degree C. 3 D. 4

8. What is the degree of homogeneity of
   $$f(x,y,z) = \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{x+y+z}.$$
   A. $\frac{1}{2}$ B. $\frac{1}{3}$ C. $-\frac{1}{2}$ D. $-\frac{1}{3}$

9. Find $\frac{dy}{dx}$ given that $y^4 + 2x^2y^2 + 6x^2 = 7$
   A. $-\frac{(xy^2+3x)}{(y^3+x^2y)}$ B. $-\frac{(xy^2+3x)}{(y^3+x^2y^2)}$ C. $-\frac{(x^2y^2+3x)}{(y^3+x^2y)}$ D. $-\frac{(x^2y+3x)}{(y^3+x^2y^2)}$

Given that $f(x, y) = \sin(xy^2)$ answer questions 10, 11, 12 and 13. Find;

10. f_x
A. $y^2 \sin(xy^2)$ B. $y^2 \cos(xy^2)$ C. $2xy\cos(xy^2)$ D. $xy^2 \cos(xy^2)$

11. f_y
A. $y^2 \sin(xy^2)$ B. $y^2 \cos(xy^2)$ C. $2xy\cos(xy^2)$ D. $xy^2 \cos(xy^2)$

12. f_{xx}
A. $-y^4 \sin(xy^2)$ B. $-y^4 \cos(xy^2)$ C. $-4xy\cos(xy^2)$ D. $-xy^2 \cos(xy^2)$

13. f_{yy}
A. $y^2 \sin(xy^2) - 4x^2y^2 \sin(xy^2)$ C. $2y\cos(xy^2) - 4x^2y^2 \sin(xy^2)$
B. $y^2 \cos(xy^2) - 4x^2y^2 \sin(xy^2)$ D. $xy^2 \cos(xy^2) - 4x^2y^2 \sin(xy^2)$

Given the spherical coordinate $\left(4, \frac{2\pi}{4}, \frac{\pi}{3}\right)$ answer questions 14 and 15.

14. Convert the spherical coordinate to Cartesian coordinate
A. $(-\sqrt{3}, 3, 2)$ B. $(\sqrt{3}, 3, 2)$ C. $(-\sqrt{3}, -3, 2)$ D. $(-\sqrt{3}, 3, -2)$

15. Convert the spherical coordinate to cylindrical coordinate
A. $\left(-2\sqrt{3}, \frac{2\pi}{3}, 2\right)$ B. $\left(2\sqrt{3}, -\frac{2\pi}{3}, 2\right)$ C. $\left(2\sqrt{3}, \frac{2\pi}{3}, -2\right)$ D. $\left(2\sqrt{3}, \frac{2\pi}{3}, 2\right)$

16. Find the equivalent cylindrical equation of the Cartesian equation $x^2 - y^2 = 25$.
A. $r^2 \cos 2\theta = 25$ B. $r\cos\theta = 25$ C. $r^2 \cos\theta = 25$ D. $r\cos 2\theta = 25$

17. Evaluate

$$\iint_R y\sin(xy)\, dA$$

Where $R = [1,2] \times [0, \pi]$.
A. 0 B. 1 C. 2 D. 3

18. If $z = f(x, y) = x^2y - 3y$ determine dz if $x = 4,\ y = 3,\ \Delta x = -0.01$ and $\Delta y = 0.02$.
A. 0.01 B. 0.02 C. 0.03 D. 0.04

19. A harmonic function of two variables satisfies
A. Laplace equation C. Poisson equation
B. Bernouli equation D. Heat equation

20. If $w = f(x, y, z)$ where $x = g(y, z)$ and $y = h(z)$, find $\left(\frac{\partial w}{\partial z}\right)_{xy}$
A. f_1 or $\frac{\partial f}{\partial x}$ B. f_2 or $\frac{\partial f}{\partial y}$ C. f_3 or $\frac{\partial f}{\partial z}$ D. f_{33} or $\frac{\partial^2 f}{\partial z^2}$

21. Given the expression $u = \sqrt{x^2 + y^2}$; where $x = re^s$ and $y = re^{-s}$. Find $\frac{\partial u}{\partial s}$
   A. $\frac{r(xe^s + ye^{-s})}{x^2+y^2}$ B. $\frac{r(xe^{-s} - ye^s)}{x^2+y^2}$ C. $\frac{r(-xe^s - ye^{-s})}{x^2+y^2}$ D. $\frac{r(xe^s - ye^{-s})}{x^2+y^2}$

22. Find the relative maximum of $f(x, y) = 2 - x^2 - xy - y^2$.
   A. (0,2) is the relative maximum point
   B. (1,0) is the relative maximum point
   C. (1,1) is the relative maximum point
   D. (0,0) is the relative maximum point

23. Find the linear approximation to the function $f(x, y, z) = xy + yz + zx$ at the point (1,1,1).
   A. $L(x, y, z) = 2x - 2y + 2z - 3$
   B. $L(x, y, z) = 2x + 2y - 2z - 3$
   C. $L(x, y, z) = 2x + 2y + 2z + 3$
   D. $L(x, y, z) = 2x + 2y + 2z - 3$

24. Let $u = x^2 + xy - y^2$ and $v = 2xy + y^2$, find $\left(\frac{\partial x}{\partial u}\right)_v$ at the point (2, −1)
   A. $\frac{1}{7}$ B. $\frac{1}{8}$ C. $\frac{1}{9}$ D. $\frac{1}{10}$

25. Two resistors in an electrical circuit with resistance R_1 and R_2 wired in parallel with a constant voltage gives an effective resistance of R, where $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. Find $\frac{\partial R}{\partial R_1}$.
   A. $\frac{R_1^2}{(R_1+R_2)^2}$ B. $\frac{R_2^2}{(R_1+R_2)^2}$ C. $\frac{R_2}{(R_1+R_2)^2}$ D. $\frac{R_1}{(R_1+R_2)^2}$

26. The partial derivative of $f(x, y, z)$ with respect of y is defined as
   A. $f_2(x, y, z) = \lim_{h_2 \to 0} \frac{f(x+h_2,y,z)+f(x,y,z)}{h_2}$
   B. $f_2(x, y, z) = \lim_{h_2 \to 0} \frac{f(x,y,z+h_2)-f(x,y,z)}{h_2}$
   C. $f_2(x, y, z) = \lim_{h_2 \to 0} \frac{f(x,y+h_2,z)+f(x,y,z)}{h_2}$
   D. $f_2(x, y, z) = \lim_{h_2 \to 0} \frac{f(x,y+h_2,z)-f(x,y,z)}{h_2}$

27. Evaluate $\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dydx$
   A. 7.28 B. 6.28 C. 5.28 D. 4.28

28. Evaluate $\int_1^4 \int_{-2}^0 \int_0^1 xyz \, dxdydz$
   A. $-\frac{15}{4}$ B. $-\frac{15}{3}$ C. $-\frac{15}{2}$ D. $-\frac{15}{1}$

29. Evaluate $\iint_R x^2 y \, dA$ where R is the region bounded by $y = 0$ and $y = 2$ for $-1 \le x \le 2$.
   A. 4 B. 5 C. 6 D. 7

30. Find the average value of the quantity $2 - x - y$ over the square $D = \{(x, y) \mid 0 \le x \le 2, 0 \le y \le 2\}$
   A. 3 B. 2 C. 1 D. 0

SOLUTION

1. Given $f(x,y) = \sin^{-1}(x+y-1)$
   For real values of $f(x,y)$
   $-1 \leq x+y-1 \leq 1$
   Answer: C

2. Given $\lim_{(x,y)\to(4,1)} \frac{xy-4y^2}{\sqrt{x}-2\sqrt{y}}$

$$\lim_{(x,y)\to(4,1)} \frac{xy-4y^2}{\sqrt{x}-2\sqrt{y}} = \lim_{(x,y)\to(4,1)} \frac{xy-4y^2}{\sqrt{x}-2\sqrt{y}} \cdot \frac{\sqrt{x}+2\sqrt{y}}{\sqrt{x}+2\sqrt{y}}$$
$$= \lim_{(x,y)\to(4,1)} \frac{y(x-4y)(\sqrt{x}+2\sqrt{y})}{x+2\sqrt{xy}-2\sqrt{xy}-4y}$$
$$= \lim_{(x,y)\to(4,1)} \frac{y(x-4y)(\sqrt{x}+2\sqrt{y})}{x-4y}$$
$$= \lim_{(x,y)\to(4,1)} y(\sqrt{x}+2\sqrt{y})$$
$$= 1(\sqrt{4}+2\sqrt{1})$$
$$= 2+2$$
$$= 4$$

   Answer: B

3. Given $\lim_{(x,y)\to(e,1)} \frac{x}{y} = \frac{e}{1} = e$
   Answer: A

4. Given $F(x,y,z,u,v) = xe^y + uz - \cos v - 2,\ G(x,y,z,u,v) = u\cos y + x^2v - yz^2 - 1$

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} F_u & G_u \\ F_v & G_v \end{vmatrix}$$
$$= \begin{vmatrix} z & \cos y \\ \sin v & x^2 \end{vmatrix}$$
$$= x^2z - \sin v \cos y \quad \text{evaluating at the point } (x,y,z,u,v) = (2,0,1,1,0)$$
$$= (2)^2(1) - \sin(0)\cos(0)$$
$$= 4(1) - (0)(1)$$
$$= 4$$

   Answer: D

5. Given $h(x,y) = \tan^{-1}(x+\sqrt{y})$
   For real values of $h(x,y)$
   $D_h = \{(x,y)\ |x \in R \ and \ y \geq 0\}$
   Answer: A

6. Given $f(x,y) = \begin{cases} x^2 + 2y & if\ (x,y) \neq (1,2) \\ 0 & if\ (x,y) = (1,2) \end{cases}$

For continuity,

$$\lim_{(x,y)\to(1,2)} f(x,y) = f(1,2)$$

$$f(1,2) = 1^2 + 2(2)$$
$$= 5$$

$$\lim_{(x,y)\to(1,2)} f(x,y) = \lim_{(x,y)\to(1,2)} f(x,y)$$
$$= \lim_{(x,y)\to(1,2)} x^2 + 2y$$
$$= 1^2 + 2(2) = 5$$

Answer: B

7. Given $f(x) = \ln x$

$f(tx) = \ln tx$

Since, $f(tx) \neq t^k \ln x$

Thus, there is no such degree.

Answer: B

8. Given $f(x,y,z) = \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{x+y+z}$

$$f(tx,ty,tz) = \frac{\sqrt{tx} + \sqrt{ty} + \sqrt{tz}}{tx + ty + tz}$$
$$= \frac{\sqrt{t}\sqrt{x}+\sqrt{t}\sqrt{y}+\sqrt{t}\sqrt{z}}{t(x+y+z)}$$
$$= \frac{\sqrt{t}(\sqrt{x}+\sqrt{y}+\sqrt{z})}{t(x+y+z)}$$
$$= \frac{t^{\frac{1}{2}}}{t}\left(\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{x+y+z}\right)$$
$$= t^{-\frac{1}{2}} f(x,y,z)$$

Thus, $-\frac{1}{2}$ is the degree of homogeneity.

Answer: C

9. Given $y^4 + 2x^2y^2 + 6x^2 = 7$

Let $F(x,y) = y^4 + 2x^2y^2 + 6x^2 - 7$

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}$$
$$= -\frac{4xy^2 + 12x}{4y^3 + 4x^2y}$$
$$= -\frac{4(xy^2 + 3x)}{4(y^3 + x^2y)}$$
$$= -\frac{(xy^2 + 3x)}{(y^3 + x^2y)}$$

Answer: A

10. Given $f(x, y) = \sin(xy^2)$

$$f_x = \frac{\partial}{\partial x} f(x, y)$$
$$= \frac{\partial}{\partial x} \sin(xy^2)$$
$$= y^2 \cos(xy^2)$$

Answer: B

11. Given $f(x, y) = \sin(xy^2)$

$$f_y = \frac{\partial}{\partial y} f(x, y)$$
$$= \frac{\partial}{\partial y} \sin(xy^2)$$
$$= 2xy \cos(xy^2)$$

Answer: C

12. Given $f(x, y) = \sin(xy^2)$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(x, y)\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} \sin(xy^2)\right)$$
$$= \frac{\partial}{\partial x}(y^2 \cos(xy^2))$$
$$= -y^4 \sin(xy^2)$$

Answer: A

13. Given $f(x, y) = \sin(xy^2)$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} f(x, y)\right)$$
$$= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} \sin(xy^2)\right)$$
$$= \frac{\partial}{\partial y}(2xy \cos(xy^2))$$
$$= 2x \cos(xy^2) - 4x^2y^2 \sin(xy^2)$$

Answer: C

14. Given $\left(4, \frac{2\pi}{3}, \frac{\pi}{3}\right) = (\rho, \theta, \phi)$

In the spherical coordinate system to Cartesian system coordinate (x, y, z)

$$x = \rho \sin\phi \cos\theta$$
$$= 4\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{2\pi}{3}\right)$$
$$= 4\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right)$$
$$= -\sqrt{3}$$

$$y = \rho \sin\phi \sin\theta$$
$$= 4\sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)$$
$$= 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$
$$= 3$$

$$z = \rho \cos\phi$$
$$= 4\cos\left(\frac{\pi}{3}\right)$$
$$= 4\left(\frac{1}{2}\right)$$
$$= 2$$

$$(x, y, z) = (-\sqrt{3}, 3, 2)$$

Answer: A

15. Given $\left(4, \frac{2\pi}{3}, \frac{\pi}{3}\right) = (\rho, \theta, \phi)$

In the spherical coordinate system to cylindrical system coordinate (r, θ, z)

$$\theta = \frac{2\pi}{3}$$
$$z = \rho\cos\phi = 4\cos\left(\frac{\pi}{3}\right) = 4\left(\frac{1}{2}\right) = 2$$
$$\rho^2 = r^2 + z^2$$
$$4^2 = r^2 + 2^2$$
$$16 = r^2 + 4$$
$$r^2 = 16 - 4$$
$$r^2 = 12$$
$$r = 2\sqrt{3}$$
$$(r, \theta, z) = \left(2\sqrt{3}, \frac{2\pi}{3}, 2\right)$$

Answer: D

16. Given $x^2 - y^2 = 25$

$$x = r\cos\theta \;\Rightarrow x^2 = r^2\cos^2\theta$$
$$y = r\sin\theta \;\Rightarrow y^2 = r^2\sin^2\theta$$
$$\Rightarrow x^2 - y^2 = r^2\cos^2\theta - r^2\sin^2\theta$$
$$\Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 25$$
$$r^2(\cos^2\theta - \sin^2\theta) = 25$$

But $\cos^2\theta - \sin^2\theta = \cos 2\theta$

Thus, $r^2\cos 2\theta = 25$

Answer: A

17. Given $\iint_R y\sin(xy)\, dA$

$$\iint^R y\sin(xy)\, dA = \int_0^{\pi}\int_1^2 y\sin(xy)\, dxdy$$
$$= \int_0^{\pi}\left[-\frac{y\cos(xy)}{y}\right]_{x=1}^{x=2} dy$$
$$= \int_0^{\pi}(-\cos(2y) + \cos y)\, dy$$
$$= \left[-\frac{\sin(2y)}{2} + \sin y\right]_0^{\pi}$$
$$= -\frac{\sin(2\pi)}{2} + \sin\pi - 0$$
$$= -\frac{0}{2} + 0$$
$$= 0$$

Answer: A

18. Given $z = x^2y - 3y,\ x = 4,\ y = 3,\ \Delta x = -0.01$ and $\Delta y = 0.02$

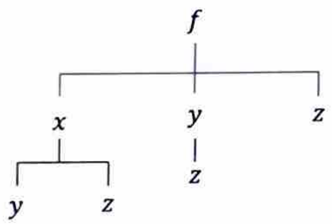
$$dz = d(x^2y - 3y)$$
$$= 2xydx + x^2dy - 3dy \quad \text{but } dx = \Delta x \text{ and } dy = \Delta y$$
$$= 2(4)(3)(-0.01) + (4)^2(0.02) - 3(0.02)$$
$$= -0.24 + 0.32 - 0.06$$
$$= 0.02$$

Answer: B

19. Laplace equation

Answer: A

20. Given $w = f(x, y, z)$ where $x = g(y, z)$ and $y = h(z)$



$$\left(\frac{\partial w}{\partial z}\right)_{xy} = f_3 \text{ or } \frac{\partial f}{\partial z}$$

Answer: B

21. Given $u = \ln\sqrt{x^2 + y^2}$; where $x = re^s$ and $y = re^{-s}$

$x = re^s$

$y = re^{-s}$

$\frac{\partial x}{\partial s} = re^s \quad \frac{\partial y}{\partial s} = -re^{-s}$

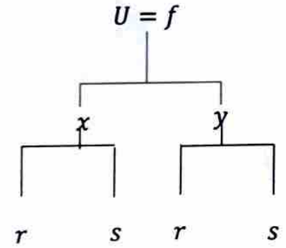
$f = \ln(x^2 + y^2)^{\frac{1}{2}} = \frac{1}{2}\ln(x^2 + y^2)$

$\frac{\partial f}{\partial x} = \frac{1}{2}\left[\frac{1}{x^2+y^2} \cdot 2x\right]$

$= \frac{x}{x^2+y^2}$

$\frac{\partial f}{\partial y} = \frac{1}{2}\left[\frac{1}{x^2+y^2} \cdot 2y\right]$

$= \frac{y}{x^2+y^2}$



$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial s}$

$= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$

$= \frac{x}{x^2+y^2} \cdot re^s + \frac{y}{x^2+y^2} \cdot -re^{-s}$

$= \frac{xre^s}{x^2+y^2} - \frac{yre^{-s}}{x^2+y^2}$

$= \frac{xre^s - yre^{-s}}{x^2+y^2}$

$= \frac{r(xe^s - ye^{-s})}{x^2+y^2}$

Answer: D

22. Given $f(x,y) = 2 - x^2 - xy - y^2$

$f_x(x,y) = -2x - y$

$f_y(x,y) = -x - 2y$

For critical points, $f_x(x,y) = f_y(x,y) = 0$

$-2x - y = 0 \qquad -x - 2y = 0$

$2x = y \qquad x = 2y$

By solving, $2x = y$ and $x = 2y$ simultaneously, we obtain

$x = 0, y = 0$

Thus, the critical point is (0,0)

Checking the nature, using Jacobian,

$$\Delta J = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}_{(0,0)}$$

$$= \begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix}_{(0,0)}$$

$$= 4 - 1$$

$$= 3 > 0$$

Also,

$$f_{xx}|_{(0,0)} = -2 < 0$$

Since, $\Delta J > 0$ and $f_{xx} < 0$ at (0,0) then the critical point (0,0) is the relative maximum point.

Answer: D

23. Given $f(x,y,z) = xy + yz + zx$

$L(x,y,z) = f(1,1,1) + f_x(1,1,1)(x-1) + f_y(1,1,1)(y-1) + f_z(1,1,1)(z-1)$

$f(1,1,1) = 1(1) + 1(1) + 1(1) = 3$

$f_x(x,y,z) = y + z \Rightarrow f_x(1,1,1) = 1 + 1 = 2$

$f_y(x,y,z) = x + z \Rightarrow f_y(1,1,1) = 1 + 1 = 2$

$f_z(x,y,z) = y + x \Rightarrow f_z(1,1,1) = 1 + 1 = 2$

$$\begin{aligned} L(x,y,z) &= 3 + 2(x-1) + 2(y-1) + 2(z-1) \\ &= 3 + 2x - 2 + 2y - 2 + 2z - 2 \\ &= 2x + 2y + 2z - 3 \end{aligned}$$

Answer: D

24. Given $u = x^2 + xy - y^2$ and $v = 2xy + y^2$

$$\frac{\partial}{\partial u}(u) = \frac{\partial}{\partial u}(x^2 + xy - y^2)$$

$$1 = 2x\frac{\partial x}{\partial u} + y\frac{\partial x}{\partial u} + x\frac{\partial y}{\partial u} - 2y\frac{\partial y}{\partial u}$$

$$1 = (2x + y)\frac{\partial x}{\partial u} + (x - 2y)\frac{\partial y}{\partial u} \ldots\ldots (1)$$

$$\frac{\partial}{\partial u}(v) = \frac{\partial}{\partial u}(2xy + y^2)$$

$$0 = 2y\frac{\partial x}{\partial u} + 2(x + y)\frac{\partial y}{\partial u}$$

$$0 = y\frac{\partial x}{\partial u} + (x + y)\frac{\partial y}{\partial u} \ldots\ldots\ldots (2)$$

Now, multiply (1) by $(x + y)$ and (2) by $(x - 2y)$, we have

$$(x + y) = (x + y)(2x + y)\frac{\partial x}{\partial u} + (x + y)(x - 2y)\frac{\partial y}{\partial u} \ldots\ldots (3)$$

$$0 = y(x - 2y)\frac{\partial x}{\partial u} + (x + y)(x - 2y)\frac{\partial y}{\partial u} \ldots\ldots\ldots (4)$$

Subtracting (4) from (3), we obtain

$$(x + y) = (x + y)(2x + y)\frac{\partial x}{\partial u} - y(x - 2y)\frac{\partial x}{\partial u}$$

$$(x + y) = (x + y)(2x + y)\frac{\partial x}{\partial u} - y(x - 2y)\frac{\partial x}{\partial u}$$

$$(x + y) = [(x + y)(2x + y) - y(x - 2y)]\frac{\partial x}{\partial u}$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{(x + y)}{(x + y)(2x + y) - y(x - 2y)}$$

Evaluating at the point $(2, -1)$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{(2 - 1)}{(2 + (-1))(2(2) + (-1)) - (-1)(2 - 2(-1))}$$

$$= \frac{1}{(2 - 1)(4 - 1) + (2 + 2)}$$

$$= \frac{1}{3 + 4}$$

$$= \frac{1}{7}$$

Answer: A

25. Given $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$
$$\frac{1}{R} = \frac{R_2 + R_1}{R_1 R_2}$$
$$R_1 R_2 = R(R_1 + R_2)$$
$$R = \frac{R_1 R_2}{R_1 + R_2}$$

$$\frac{\partial R}{\partial R_1} = \frac{\partial}{\partial R_1}\left(\frac{R_1 R_2}{R_1 + R_2}\right)$$
$$= \frac{(R_1 + R_2)(R_2) - R_1 R_2(1)}{(R_1 + R_2)^2}$$
$$= \frac{R_1 R_2 + R_2^2 - R_1 R_2}{(R_1 + R_2)^2}$$
$$= \frac{R_2^2}{(R_1 + R_2)^2}$$

Answer: B

26. The partial derivative of $f(x, y, z)$ with respect of y is defined as

$$f_2(x, y, z) = \lim_{h_2 \to 0} \frac{f(x, y + h_2, z) - f(x, y, z)}{h_2}$$

Answer: D

27. Given $\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dydx$

$$\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dydx = \int_1^4 \int_1^2 \left(x \cdot \frac{1}{y} + y \cdot \frac{1}{x}\right) dydx$$
$$= \int_1^4 \left[x \ln y + \frac{1}{x} \cdot \frac{y^2}{2}\right]_{y=1}^{y=2} dx$$
$$= \int_1^4 \left[x \ln 2 + \frac{1}{x} \cdot \frac{2^2}{2}\right] - \left[x \ln 1 + \frac{1}{x} \cdot \frac{1^2}{2}\right] dx$$
$$= \int_1^4 \left[x \ln 2 + 2 \cdot \frac{1}{x}\right] - \left[x(0) + \frac{1}{2} \cdot \frac{1}{x}\right] dx$$
$$= \int_1^4 \left[x \ln 2 + 2 \cdot \frac{1}{x} - \frac{1}{2} \cdot \frac{1}{x}\right] dx$$

$$= \int_1^4 \left[x \ln 2 + \frac{3}{2} \cdot \frac{1}{x}\right] dx$$

$$= \left[\frac{x^2}{2} \ln 2 + \frac{3}{2} \ln x\right]_{x=1}^{x=4}$$

$$= \left[\frac{4^2}{2} \ln 2 + \frac{3}{2} \ln 4\right] - \left[\frac{1^2}{2} \ln 2 + \frac{3}{2} \ln 1\right]$$

$$= \left[8 \ln 2 + \frac{3 \ln 4}{2}\right] - \left[\frac{\ln 2}{2} + \frac{3}{2}(0)\right]$$

$$= 5.55 + 2.08 - 0.35 - 0$$

$$= 7.28$$

Answer: A

28. Given $\int_1^4 \int_{-2}^0 \int_0^1 xyz \, dxdydz$

$$\int_1^4 \int_{-2}^0 \int_0^1 xyz \, dxdydz = \int_1^4 \int_{-2}^0 \left[\frac{x^2}{2} \cdot yz\right]_{x=0}^{x=1} dydz$$

$$= \int_1^4 \int_{-2}^0 \frac{1}{2} \cdot yz \, dydz$$

$$= \frac{1}{2} \int_1^4 \left[\frac{y^2}{2} z\right]_{y=-2}^{y=0} dz$$

$$= \frac{1}{2} \int_1^4 (0 - 2z) \, dz$$

$$= \frac{1}{2} \int_1^4 (-2z) \, dz$$

$$= \frac{1}{2} \left[-2 \cdot \frac{z^2}{2}\right]_{z=1}^{z=4}$$

$$= -\left[\frac{z^2}{2}\right]_{z=1}^{z=4}$$

$$= -\left[\frac{4^2}{2} - \frac{1^2}{2}\right]$$

$$= -\left(8 - \frac{1}{2}\right)$$

$$= -\frac{15}{2}$$

Answer: C

29. Given $\iint_R x^2y\,dA$ $y = 0, y = 2$ and $-1 \le x \le 2$

$$\iint_R x^2y\,dA = \int_{-1}^{2}\int_0^2 x^2y\,dydx$$
$$= \int_{-1}^{2}\left[\frac{x^2y^2}{2}\right]_{y=0}^{y=2} dx$$
$$= \int_{-1}^{2}\frac{x^2 2^2}{2} - 0\,dx$$
$$= \int_{-1}^{2} 2x^2\,dx$$
$$= \left[\frac{2x^3}{3}\right]_{x=-1}^{x=2}$$
$$= \frac{2(2)^3}{3} - \frac{2(-1)^3}{3}$$
$$= \frac{16}{3} + \frac{2}{3}$$
$$= 6$$

Answer: C

30. Given $2 - x - y$ over the square $D = \{(x, y) \mid 0 \le x \le 2,\ 0 \le y \le 2\}$

$$\int_0^2\int_0^2 (2 - x - y)\,dydx = \int_0^2\left[2y - xy - \frac{y^2}{2}\right]_{y=0}^{y=2} dx$$
$$= \int_0^2\left(2(2) - 2x - \frac{2^2}{2} - 0\right)dx$$
$$= \int_0^2 (2 - 2x)\,dx$$
$$= \left[2x - \frac{2x^2}{2}\right]_{x=0}^{x=2}$$
$$= 2(2) - (2)^2$$
$$= 4 - 4$$
$$= 0$$

Answer: D